Local exact controllability of the 2D-Schrödinger-Poisson system

Karine Beauchard, Camille Laurent

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Abstract

In this article, we investigate the exact controllability of the $2D$-Schrödinger-Poisson system, which couples a Schrödinger equation on a bounded domain of $\mathbb{R}^2$ with a Poisson equation for the electrical potential. The control acts on the system through a Neumann boundary condition on the potential, locally distributed on the boundary of the space domain. We prove several results, with or without nonlinearity and with different boundary conditions on the wave function, of Dirichlet type or of Neumann type.

1 Introduction

Microelectronics industry has driven transistor sizes to the nanometer scale. This has led to the possibility of building nanostructures like single electron transistors or single electron memories, which involve the transport of only a few electrons. In general, such devices consist in an active region, on which the electrical potential can be tuned by an electrode (the gate). In many applications, the performance of the device will depend on the possibility of controlling the electrons by acting on the gate voltage.

At the nanometer scale, quantum effects become important and a quantum transport model is necessary. In this paper, we analyze the controllability of a simplified mathematical model, of the quantum transport of electrons trapped in a two-dimensional device. The model consists in a single Schrödinger equation, on a 2D bounded domain, coupled to the Poisson equation for the electrical potential. The control acts on the system through a Neumann boundary condition on the pontential, on a part of the boundary, modelling the gate. Such a model has already been studied in [26] for controllability purposes.

It would be physically relevant to impose Dirichlet boundary conditions on the wave function and to take into account the self-consistent potential modeling interactions between electrons. However, the mathematical analysis of this configuration is quite complicated, thus we investigate two simpler configurations:

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1. a first configuration, in which Dirichlet boundary conditions are imposed on the wave function, but we neglect the self-consistent potential,

2. a second configuration, in which we take into account the self-consistent potential, but Neumann boundary conditions are imposed on the wave function.

This work is a first step towards more realistic models.

1.1 Linear PDE with Dirichlet boundary conditions

First, we consider the system

$$\begin{cases} \Big(i\partial_t + \Delta\Big)\psi(t,x) = v(t,x)\psi(t,x)\,, & (t,x) \in (0,T) \times \Omega\,, \\ \psi(t,x) = 0\,, & (t,x) \in (0,T) \times \partial\Omega\,, \\ \psi(0,x) = \psi_0(x) & x \in \Omega\,, \\ (-\Delta + 1)v(t,x) = 0\,, & (t,x) \in (0,T) \times \Omega\,, \\ \partial_\nu v(t,x) = g(t,x)1_{\Gamma_c}(x) & (t,x) \in (0,T) \times \partial\Omega\,, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^2$, Γ_c is an open subset of $\partial\Omega$ and 1_{Γ_c} is its characteristic function. The control is the real valued function g and we want to control the wave function ψ.

1.1.1 Definitions and notations

We denote by $\langle .,. \rangle$ the complex valued scalar product on $L^2(\Omega,\mathbb{C})$,

$$\langle f, g\rangle := \int_\Omega f(x)\overline{g(x)}dx\,,$$

by $\mathcal{S}$ the $L^2(\Omega,\mathbb{C})$-sphere

$$\mathcal{S} := \{\xi \in L^2(\Omega,\mathbb{C}); \|\xi\|_{L^2(\Omega)} = 1\}\,,$$

by $(-\Delta_D)$ the Laplace operator associated with Dirichlet boundary conditions

$$D(-\Delta_D) := H^2 \cap H^1_0(\Omega,\mathbb{C})\,, \quad -\Delta_D\xi := -\Delta\xi$$

by $(\lambda_k)_{k\in\mathbb{N}^*}$ the nondecreasing sequence of its eigenvalues, by $(\varphi_k)_{k\in\mathbb{N}^*}$ the associated normalized eigenfunctions,

$$\begin{cases} -\Delta\varphi_k(x) = \lambda_k\varphi_k(x)\,, & x \in \Omega\,, \\ \varphi_k(x) = 0\,, & x \in \partial\Omega\,, \\ \|\varphi_k\|_{L^2(\Omega)} = 1\,, \end{cases}$$

by $H^3_{(0)}(\Omega)$ the Sobolev space

$$H^3_{(0)}(\Omega) := D\Big((-\Delta_D)^{3/2}\Big) = \{\xi \in H^3 \cap H^1_0(\Omega,\mathbb{C}); \Delta\xi \in H^1_0(\Omega,\mathbb{C})\}\,,$$

and by $\mathbb{P}_K$, for $K \in \mathbb{N}^*$, the projection

$$\left|\begin{array}{lccl} \mathbb{P}_K : & H^{-1}(\Omega,\mathbb{C}) & \to & \mathrm{Adh}_{H^{-1}(\Omega)}\left(\mathrm{Span}(\varphi_k; k \geqslant K)\right) \\ & \xi & \mapsto & \xi - \sum_{k=1}^{K-1}\langle \xi, \varphi_k\rangle_{H^{-1},H^1_0}\,\varphi_k \end{array}\right.$$

where $\langle .,. \rangle_{H^{-1},H^1_0}$ is the duality product between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. We also use the weak observability of the Schrödinger equation, defined below; several configurations (Ω,Γ_c) for which it has been proved are recalled in Section 2.1.

Definition 1 (Observability and weak observability of Schrödinger equation). *Let Γ be an open subset of $\partial\Omega$ and $0 \leqslant \tau_1 < \tau_2 < \infty$. The Schrödinger equation on Ω is observable (resp. weakly observable) on $(\tau_1, \tau_2) \times \Gamma$ if there exists $\mathcal{C}_0 > 0$ such that*

$$\|\phi_T\|_{H^1_0(\Omega)} \leqslant \mathcal{C}_0 \|\partial_\nu \phi\|_{L^2((\tau_1,\tau_2)\times\Gamma)}\,, \quad \forall \phi_T \in H^1_0(\Omega, \mathbb{C})$$

$$\Big(\textit{resp. } \|\phi_T\|_{H^1_0(\Omega)} \leqslant \mathcal{C}_0 \Big(\|\partial_\nu \phi\|_{L^2((\tau_1,\tau_2)\times\Gamma)} + \|\phi_T\|_{H^{-1}(\Omega)} \Big)\,, \quad \forall \phi_T \in H^1_0(\Omega, \mathbb{C}) \Big) \tag{2}$$

where $\phi(t) := e^{i\Delta_D (t-T)}\phi_T$, i.e. ϕ is the solution of

$$\left\{ \begin{array}{ll} \Big(i\partial_t + \Delta\Big)\phi(t,x) = 0\,, & (t,x) \in (0,T) \times \Omega\,, \\ \phi(t,x) = 0\,, & (t,x) \in (0,T) \times \partial\Omega\,, \\ \phi(T,x) = \phi_T(x)\,, & x \in \Omega\,. \end{array} \right. \tag{3}$$

The Schrödinger equation on Ω is observable (resp. weakly observable) on Γ if it is observable (resp. weakly observable) on $(\tau_1, \tau_2) \times \Gamma$ for some $0 \leqslant \tau_1 < \tau_2 < \infty$.

In the inequality (2), the last term $\|\phi_T\|_{H^{-1}(\Omega)}$ may be equivalently replaced by any term of the form $\|\phi_T\|_{H^s(\Omega)}$ with $-1 < s < 1$, by an interpolation argument.

1.1.2 Local exact controllability of high frequencies

Our first result is the local exact controllability of high frequencies of system (1), around any trajectory of the free system.

Theorem 1. *Let Ω be a bounded open subset of $\mathbb{R}^2$, Γ_c be a non empty open subset of $\partial\Omega$, $T > 0$, $\psi_0 \in H^3_{(0)}(\Omega) \cap \mathcal{S}$ and $\psi_{ref}(t) := e^{i\Delta_D t}\psi_0$. We assume that*

- ***[H1]:** either Ω is of class C^∞ and locally on one side of $\partial\Omega$; or $\Omega = (0,\pi) \times (0,L)$ for some $L > 0$ and $\overline{\Gamma_c}$ does not contain any vertex of Ω,*
- ***[H2]:** the Schrödinger equation on Ω is weakly observable on $(0, \widetilde{T}) \times \Gamma_c$ for some $\widetilde{T} \in (0,T)$,*
- ***[H3]:** $|\partial_\nu \psi_{ref}(t,x)| \geqslant m > 0\,, \quad \forall (t,x) \in (T', T'') \times \Gamma_c$ for some $T', T'' \in [0,T]$ such that $T'' - T' > \widetilde{T}$.*

Then, there exists $K, \delta > 0$ and a C^1-map

$$\Upsilon : \mathcal{V} \to L^2((0,T) \times \Gamma_c, \mathbb{R})$$

where $\mathcal{V} := \{\psi_f \in \mathbb{P}_K[H^3_{(0)}(\Omega, \mathbb{C})];\, \|\psi_f - \mathbb{P}_K[\psi_{ref}(T)]\|_{H^3_{(0)}} < \delta\}$, such that

- *$\Upsilon(\mathbb{P}_K[\psi_{ref}(T)]) = 0$,*
- *for every $\psi_f \in \mathcal{V}$, the solution of (1) with control $g = \Upsilon(\psi_f)$ satisfies $\mathbb{P}_K[\psi(T)] = \psi_f$.*

As a consequence, there exists $K' \geqslant K$ and $g \in L^2((0,T) \times \Gamma_c, \mathbb{R})$ such that the solution of (1) satisfies $\mathbb{P}_{K'}[\psi(T)] = 0$; in particular, $x \mapsto \psi(T,x)$ is a smooth function.

Such a result may be used to prove global exact controllability of the Schrödinger-Poisson system in $H^3_{(0)}(\Omega, \mathbb{C})$ with controls $g \in L^2((0,T), \mathbb{R})$, by following the strategy of [6]. This will be at the core of future works by the authors.

In particular, Theorem 1 applies, for arbitrary $T > 0$ and $\psi_0 \in H^3_{(0)}(\Omega) \cap \mathcal{S}$, when (see Propositions 1 and 2)

- $\Omega = (0, \pi) \times (0, L)$ for some $L > 0$ and Γ_c contains both a horizontal and a vertical segment,
- Ω is a disk and Γ_c is arbitrary.

Theorem 1 also applies when (Ω, Γ_c) satisfies the Geometric Control Condition, for any $T > 0$ and $\psi_0 \in H^3_{(0)}(\Omega) \cap \mathcal{S}$ that satisfy **[H3]** (see Proposition 1).

Assumption **[H1]** is important for system (1) to be well-posed in $H^3_{(0)}(\Omega)$ with control $g \in L^2((0,T) \times \Gamma_c, \mathbb{R})$ (see Section 3.4). When Ω is a rectangle, it is important to assume that its vertices do not belong to $\overline{\Gamma_c}$, in order to take advantage of the usual elliptic regularity on the potential v.

1.1.3 Local exact controllability around an eigenstate

Our second result is the local exact controllability of system (1) around an eigenstate.

Theorem 2. *Let Ω be a bounded open subset of $\mathbb{R}^2$, Γ_c be an open subset of $\partial\Omega$, $T > 0$, $R \in \mathbb{N}^*$ and $\psi_{ref}(t) := \varphi_R(x) e^{-i\lambda_R t}$. We assume **[H1]**, **[H2]**,*

- ***[H3']:** $|\partial_\nu \varphi_R(x)| \geqslant m > 0\,, \forall x \in \Gamma_c$,*
- ***[H4]:** λ_R is a simple eigenvalue of $(-\Delta_D)$ and for any eigenvector Φ of $(-\Delta_D)$, the solution w of*

$$\begin{cases} (-\Delta + 1) w(x) = \varphi_R(x) \Phi(x)\,, & x \in \Omega\,, \\ \partial_\nu w(x) = 0\,, & x \in \partial\Omega\,, \end{cases}$$

  does not identically vanish on Γ_c: $w \neq 0$ on Γ_c.

Then, there exists $\delta > 0$ and a C^1-map

$$\Upsilon : \mathcal{V} \to L^2((0,T) \times \Gamma_c, \mathbb{R})$$

where $\mathcal{V} := \{(\psi_0, \psi_f) \in [H^3_{(0)}(\Omega, \mathbb{C}) \cap \mathcal{S}]^2; \|\psi_0 - \psi_{ref}(0)\|_{H^3_{(0)}} + \|\psi_f - \psi_{ref}(T)\|_{H^3_{(0)}} < \delta\}$, such that $\Upsilon[\psi_{ref}(0), \psi_{ref}(T)] = 0$ and, for every $(\psi_0, \psi_f) \in \mathcal{V}$, the solution of (1) with control $g = \Upsilon(\psi_0, \psi_f)$ satisfies $\psi(T) = \psi_f$.

Note that, under assumption **[H1]**, then assumption **[H3']** systematically holds with $R = 1$. The assumptions of Theorem 2 could look quite technical, but roughly speaking the main assumptions are:

- the weak observability (Assumption **[H2]**), that ensures the controllability of high frequencies, as in Theorem 1,
- the unique continuation property (Assumption **[H4]**), that ensures the controllability of low frequencies, see Proposition 5).

If **[H2]** holds, but **[H4]** is not satisfied, then the linearised system around ψ_{ref} is not controllable: it misses a finite number of directions corresponding to the eigenfunctions Φ for which **[H4]** is not satisfied, see Remark 2. It would be interesting to recover these directions by power series expansions [17, Chapter 8].

Theorem 2 applies, in particular, when $\Omega = (0,\pi)\times(0,L)$ for some $L>0$, Γ_c is an open subset of $\partial\Omega$ that contains both a horizontal and a vertical segment and λ_R is a simple eigenvalue of $(-\Delta_D)$ (see Proposition 3).

The validity of Theorem 2 on the disk $\Omega = \{(x,y) \in \mathbb{R}^2; x^2+y^2<1\}$ is an open problem: cheking Assumption **[H4]** would require some study on the product of Bessel functions, which might require lengthy computations.

Following arguments by Méhats, Privat and Sigalotti [26] (relying on methods of Privat-Sigallotti [33, 32]), it should be possible to prove that **[H4]** holds generically with respect to the domain Ω. This becomes more clear with some equivalent ways of expressing Assumption **[H4]** described in Proposition 4.

Assumption **[H4]** is quite unusual in control theory. It looks like a classical unique continuation property for eigenfunctions but it does not seem that we can refer to some already known results to prove it in great generality. We have indeed chosen to discuss about it more precisely in Subsection 2.3.

In principle, it may be possible to state a Theorem of exact controllability similar to Theorem 2 close to any reference trajectory ψ_{ref} as done in Theorem 1. Yet, the equivalent of Assumption **[H4]** would be a unique continuation type assumption complicated to state and depending on time. That is why we have chosen to state Theorem 2 only close to an eigenfunction where the unique continuation type Assumption takes the nice form of **[H4]**.

1.2 Controllability of Schrödinger equation with real valued control

For the Schrödinger-Poisson system, the control g corresponds to a potential and therefore needs to be real valued to have a physical meaning. Therefore, as an intermediary result, we will also get the exact controllability of Schrödinger equation with *real valued* controls (instead of complex valued ones in the existing literature), when the equation is weakly observable. The result that we need for the control of the Schrödinger-Poisson system will actually be more complicated. Yet, we believed that it could be useful for other contexts and we give a simpler proof in the simpler context of the free Schrödinger equation (see for instance Araruna-Cerpa-Mercado-Santos [2] where related questions are raised).

Theorem 3. *Let Ω be an open subet of $\mathbb{R}^2$, Γ be an open subset of $\partial\Omega$ and $0 < \widetilde{T} < T < \infty$. If the Schrödinger equation on Ω is weakly observable on $(0,\widetilde{T})\times\Gamma$ then, for every $\psi_f \in H^{-1}(\Omega,\mathbb{C})$, there exists a **real valued** control*

$u \in L^2((0,T) \times \partial\Omega, \mathbb{R})$ *such that the solution of*

$$\begin{cases} (i\partial_t + \Delta)\,\psi = 0\,, & (t,x) \in (0,T) \times \Omega\,, \\ \psi(t,x) = u(t,x)1_\Gamma(x)\,, & (t,x) \in (0,T) \times \partial\Omega\,, \\ \psi(0,x) = 0\,, & x \in \Omega\,, \end{cases}$$

satisfies $\psi(T) = \psi_f$.

The proof will be given in Section 6.

1.3 Nonlinear PDE, on a rectangle, with Neumann boundary conditions

Now, we consider the nonlinear Schrödinger-Poisson system on a rectangle

$$\begin{cases} i\partial_t\psi(t,x) = -\Delta\psi(t,x) + \widetilde{v}(t,x)\psi(t,x)\,, & (t,x) \in (0,T) \times \Omega\,, \\ \partial_\nu\psi(t,x) = 0\,, & (t,x) \in (0,T) \times \partial\Omega\,, \\ \psi(0,x) = \psi_0(x)\,, & x \in \Omega\,, \\ (-\Delta + 1)\widetilde{v}(t,x) = \epsilon|\psi(t,x)|^2\,, & (t,x) \in (0,T) \times \Omega\,, \\ \partial_\nu\widetilde{v}(t,x) = g(t,x)1_{\Gamma_c}(x)\,, & (t,x) \in (0,T) \times \partial\Omega\,, \end{cases} \tag{4}$$

where $\epsilon \in \mathbb{R}$, $x = (x_1, x_2) \in \Omega := (0,\pi) \times (0,L)$, $L > 0$ and Γ_c is an open subset of $\partial\Omega$. The control is the real valued function g and we want to control the wave function ψ.

For the nonlinear system (4), our main result is the local exact controllability around the reference trajectory, constant in space,

$$\left(\psi_{ref}(t,x) = \frac{e^{-i\frac{\epsilon t}{\sqrt{\pi L}}}}{\sqrt{\pi L}}\,, \widetilde{v}_{ref}(t,x) = 0\,, g_{ref}(t,x) = 0\right)\,. \tag{5}$$

We denote by Δ_N the Laplace operator associated with Neumann boundary conditions

$$\begin{aligned} &D(\Delta_N) = H^2_N(\Omega,\mathbb{C}) := \{\varphi \in H^2(\Omega,\mathbb{C}); \partial_\nu\varphi = 0 \text{ on } \partial\Omega\} \\ &\Delta_N\varphi := \Delta\varphi\,. \end{aligned}$$

Theorem 4. *Let* $L > 0$, $\Omega := (0,\pi) \times (0,L)$, Γ_c *be an open subset of* $\partial\Omega$ *such that* $\overline{\Gamma_c}$ *does not contain any vertex of* Ω, $T > 0$, $\epsilon \in \mathbb{R}$ *be such that*

$$\epsilon > -\frac{\pi L}{2} m(m+1)^2 \quad \text{where} \quad m := \min\left\{1; \left(\frac{\pi}{L}\right)^2\right\}\,. \tag{6}$$

and ψ_{ref} *be defined by (5). There exists* $\delta > 0$ *and a* C^1*-map*

$$\Upsilon : \mathcal{V} \to L^2((0,T) \times \partial\Omega, \mathbb{R})$$

where

$$\mathcal{V} := \left\{(\psi_0,\psi_f) \in [H^2_N(\Omega,\mathbb{C}) \cap \mathcal{S}]^2; \|\psi_0 - \psi_{ref}(0)\|_{H^2} + \|\psi_f - \psi_{ref}(T)\|_{H^2} < \delta\right\}$$

such that $\Upsilon(\psi_{ref}(0), \psi_{ref}(T)) = 0$ *and for every* $(\psi_0,\psi_f) \in \mathcal{V}$, *the solution of (4) with control* $g = \Upsilon(\psi_0,\psi_f)$ *satisfies* $\psi(T) = \psi_f$.

This part with the nonlinear Schrödinger equation only deals with Ω a rectangle, where the boundary is flat. Indeed, with the Neumann boundary control, the smoothing effect, required in our proof, is not well understood for a general open set Ω. The nonhomogeneous boundary Cauchy problem is then quite complicated. This was for instance investigated for the wave equation by Tataru [35] and earlier papers by Lasiecka and Triggiani [20]. The curvature of the boundary has important consequences in this case.

1.4 Bibliography

1.4.1 Schrödinger equation with bilinear control

The Schrödinger equation with bilinear control has been widely studied in the litterature, and is related to the system under study in this article. This model writes

$$\begin{cases} (i\partial_t + \Delta - V)\psi(t,x) = u(t)\mu(x)\psi(t,x)\,, & (t,x) \in (0,T) \times \Omega\,, \\ \psi(t,x) = 0\,, & (t,x) \in (0,T) \times \partial\Omega\,, \end{cases} \tag{7}$$

where $V, \mu : \Omega \to \mathbb{R}$ are given functions, the state ψ lives in $\mathcal{S}$ and the control is the real valued function $u : (0,T) \to \mathbb{R}$. When V and μ solve an appropriate Poisson equation, then system (7) is a particular case of Schrï¿$\frac{1}{2}$dinger-Poisson system (1) (take $v(t,x) = V(x) + u(t)\mu(x)$).

A negative result A negative control result was proved by Turinici in [37], as a consequence of a general result by Ball, Marsden and Slemrod in [3]. It states that, for $V = 0$, for a given function $\mu \in C^2(\Omega, \mathbb{R})$, for a given initial condition $\psi_0 \in (H^2 \cap H^1_0)(\Omega, \mathbb{C}) \cap \mathcal{S}$, and by using controls $u \in L^r_{loc}((0,\infty), \mathbb{R})$ with $r > 1$, one may only reach a subset of $(H^2 \cap H^1_0)(\Omega) \cap \mathcal{S}$ that has an empty interior in $(H^2 \cap H^1_0)(\Omega, \mathbb{C}) \cap \mathcal{S}$. Recently, Boussaid, Caponigro and Chambrion extended this negative result to the case of controls in $L^1_{loc}((0,\infty), \mathbb{R})$, see [11]. However, this negative results are actually due to an inappropriate choice of functional setting, as emphasized in the next paragraph.

Note that this type of negative results is specific to systems with a bilinear control, which is a function ot the time variable t (and not a function of both t and x)). Thus, it does not apply to the Schrödinger-Poisson system (1) studied in this article. See also Section 5.3 for some comments about this type of control in our context.

Local exact results in 1D Beauchard proved in [4] the exact controllability of equation (7), locally around the ground state in H^7, with controls $u \in H^1((0,T), \mathbb{R})$ in large time T, in the case $N = 1$, $\Omega = (-1/2, 1/2)$, $\mu(x) = x$ and $V = 0$. The proof of [4] relies on Coron's return method and Nash-Moser theorem.

Reference [5], by the authors of this paper, improves this result and establishes the exact controllability of equation (7), locally around the ground state in H^3, with controls $u \in L^2((0,T), \mathbb{R})$, in arbitrary time $T > 0$, and with generic functions μ when $N = 1$, $\Omega = (0,1)$. This result, proved with $V = 0$ in [5], can be extended to an arbitrary potential V, as explained in [25]. The proof relies on a smoothing effect, that allows to conclude with the inverse mapping theorem (instead of Nash-Moser's one).

Then, Morancey and Nersesyan developped this stategy to control a Schrödinger equation with a polarizability term [24] and a finite number of Schrödinger equations with one control [23, 25].

The goal of this article is to extend this type of strategy (previously applied in 1D) to the 2D Schrödinger-Poisson system (1). New difficulties appear at the 2 levels of the proof:

- in controlling the linearized system, because a new observability inequality is required,
- in proving the appropriate smoothing effect or the 2D-system (1).

Global approximate results Three strategies have been developed to study approximate controllability for equation (7)

The first strategy is a variationnal argument introduced by Nersesyan in [29]. It proves the global controllability to the ground state, approximately in H^3, with smooth controls $u \in C_c^\infty((0,T),\mathbb{R})$, in large time T, for generic functions (μ, V), in arbitray dimension N.

Note that this global approximate control result may be coupled to the previous local exact controllability results to provide global exact controllability, see [30] for equation (7), [24] for a Schrödinger equation with a polarizability term, [25] for finite number of Schrödinger equations with the same control.

A second strategy consists in deducing approximate controllability in regular spaces (containing H^3) from exact controllability results in infinite time by Nersesyan and Nersisyan [31]

A third strategy, due to Chambrion, Mason, Sigalotti, and Boscain [16], relies on geometric techniques for the controllability of the Galerkin approximations. It proves (under appropriate assumptions on V and μ) the approximate controllability of (7) in L^2, with piece-wise constant controls. The hypotheses of this result were refined by Boscain, Caponigro, Chambrion, and Sigalotti in [8]. The approximate controllability is proved in higher Sobolev norms in [11] for one equation, and in [10] for a finite number of equations with one control. For more details and more references about the geometric techniques, we refer the reader to the recent survey [9].

1.4.2 Schrödinger-Poisson system

In [26], Méhats, Privat and Sigalotti prove the approximate controllability in L^2 for a Schrödinger-Poisson system, with mixed boundary conditions (of Dirichlet or Neumann type, depending on the place where we are on the boundary). This result holds for generic domains Ω and generic control supports Γ_c on the boundary. The proof relies on the general result of [16] and analyticity arguments to obtain genericity.

1.5 Structure of the article

This article is organized in 7 Sections.

Section 2 is a discussion about our assumptions **[H2]**, **[H3]** and **[H4]**, in particular when Ω is a rectangle domain or a disk.

Section 3 is dedicated to the well-posedness of system (1), in appropriate spaces for our control problem. The starting point is a smoothing effect proved by Puel in [34], that needs to be recast in our context.

In Section 4, we prove local exact controllability of hight frequencies, around any trajectory of the free system, i.e. Theorem 1.

Section 5 is devoted to the proof of the local exact controllability, around any eigenstate, i.e. Theorem 2.

The goal of Section 6 it to explain how the strategy of Section 5 can be used to get controllability for the Schrödinger equation with *real-valued* boundary controls (instead of complex valued ones in the literature), i.e. Theorem 3.

Finally, in Section 7, we prove the local exact controllability of the nonlinear Schrï¿½dinger-Poisson system (4) on a rectangle, i.e. Theorem 4.

1.6 Notations

Implicitly functions take values in $\mathbb{C}$, otherwise we specify, for instance $L^2((0,T)\times\partial\Omega,\mathbb{R})$. The volume element on Ω is denoted dx and the surface element on $\partial\Omega$ is denoted $d\sigma(x)$. When $\varphi\in\mathcal{S}$ then $T_{\mathcal{S}}\varphi$ denotes the tangent space to the sphere $\mathcal{S}$ at point φ

$$T_{\mathcal{S}}\varphi := \left\{\xi\in L^2(\Omega,\mathbb{C}); \Re\left(\int_\Omega \varphi(x)\overline{\xi(x)}dx\right)=0\right\}.$$

2 Discussion on our assumptions

The goal of this section is to prove that our assumptions **[H2]**, **[H3]** and **[H4]** hold for appropriate configurations (Ω,Γ_c) and to explain why **[H4]** is related to the control of low frequencies.

2.1 Weak observability of Schrödinger equation [H2]

The goal of this section is to recall known results about assumption **[H2]**.

Proposition 1. *Property **[H2]** holds in the following cases.*

1. *$\Omega=(0,\pi)\times(0,L)$ for some $L>0$, $\widetilde{T}>0$ and Γ_c is an open subset of $\partial\Omega$ that contains both a horizontal and a vertical segment with nonzero length (necessary and sufficient condition).*

2. *Ω is smooth, does not have a contact of infinite order with its tangents and (Ω,Γ_c) satisfies the Geometric Control Condition.*

3. *$\Omega=\{(x,y)\,|x^2+y^2<1\}$ and Γ_c is an arbitrary open set of the boundary.*

Statement 1 is proved by Tenenbaum and Tucsnak in [36, Theorem 1.4]. Precisely, they prove that the Schrödinger equation is observable on $(0,\widetilde{T})\times\Gamma_c$ iff Γ_c contains both a horizontal and a vertical segment with nonzero length. Thus weak observability also holds in this configuration. Moreover, the same counter-example as in [36, page 967] proves that this condition is necessary: with $\Gamma_c:=(a,b)\times\{0\}$, $0\leqslant a<b\leqslant\pi$, $m\in\mathbb{N}^*$ and $\phi_T(x,y):=\sin(mx)\sin(\pi y/L)$, we get

$$Cm\leqslant\|\phi_T\|_{H^1_0(\Omega)} \quad\text{and}\quad \|\partial_\nu\phi\|_{L^2((T',T'')\times\Gamma_c)}+\|\phi_T\|_{H^{-1}(\Omega)}\leqslant C',$$

for some constants $C, C' > 0$ independant of m.

Statement 2 is proved by Lebeau in [21] when Ω is analytic. The result is actually also true if Ω is smooth (see for instance Burq-Zworski [15]). The assumptions of contact of finite order is made in order to ensure the uniqueness of the broken geodesic flow. It is certainly also true if Ω is only C^3 using the result by Burq [13] for the wave equation and transmutation or resolvent methods (see Miller [27, 28] for instance).

Statement 3 is proved by Anatharaman, Léautaud and Macià in [1].

2.2 Simulatenous validity of [H2] and [H3]

The goal of this section is to prove the claim written after Theorem 1.

Proposition 2. *We assume that*

- *either Ω is a disk and Γ_c is a non empty open subset of $\partial\Omega$,*
- *or $\Omega = (0, \pi) \times (0, L)$ for some $L > 0$ and Γ_c contains both a horizontal and a vertical segment.*

*Let $T > 0$, $\psi_0 \in H^3_{(0)}(\Omega, \mathbb{C}) \cap \mathcal{S}$ and $\psi_{ref}(t) := e^{i\Delta_D t}\psi_0$. Then assumptions **[H2]** and **[H3]** are satisfied when Γ_c is replaced by an appropriate nonempty open subset Γ_c' of Γ_c. As a consequence, the conclusion of Theorem 1 holds.*

Proof of Proposition 2: Let us assume that Ω is a disk and Γ_c is a non empty open subset of $\partial\Omega$. By unique continuation for the Schrödinger equation, we know that the continuous function $\partial_\nu\psi_{ref}$ does not identically vanish on $(0, T) \times \Gamma_c$. Thus, there exists $0 \leqslant T' < T'' \leqslant T$ and an open subset Γ_c' of Γ_c such that $|\partial_\nu\psi_{ref}(t, x)| \geqslant m > 0$ for every $(t, x) \in (T', T'') \times \Gamma_c'$. Let $\widetilde{T} \in (0, T'' - T')$. By the previous proposition, the Schrödinger equation on Ω is weakly observable on $(0, \widetilde{T}) \times \Gamma_c'$. Therefore, both **[H2]** and **[H3]** hold with Γ_c replaced by Γ_c'. Then, Theorem 1 provides controls supported in $(0, T) \times \Gamma_c'$ that are, a fortiori, also supported in $(0, T) \times \Gamma_c$.

Now, let us assume that $\Omega = (0, \pi) \times (0, L)$ for some $L > 0$ and Γ_c contains both a horizontal segment Γ_H and a vertical segment Γ_V. By unique continuation for the Schrödinger equation, we know that $\partial_\nu\psi_{ref}$ does not identically vanish on $(0, T) \times \Gamma_H$. Thus, there exists $0 \leqslant T_H' < T_H'' \leqslant T$ and an open subset Γ_H' of Γ_H such that $|\partial_\nu\psi_{ref}(t, x)| \geqslant m_H > 0$ for every $(t, x) \in (T_H', T_H'') \times \Gamma_H'$. By unique continuation for the Schrödinger equation, we know that $\partial_\nu\psi_{ref}$ does not identically vanish on $(T_H', T_H'') \times \Gamma_V$. Thus, there exists $T_H' \leqslant T' < T'' \leqslant T_H''$ and an open subset Γ_V' of Γ_V such that $|\partial_\nu\psi_{ref}(t, x)| \geqslant m_V > 0$ for every $(t, x) \in (T', T'') \times \Gamma_V'$. Then $|\partial_\nu\psi_{ref}(t, x)| \geqslant m := \min\{m_H; m_V\} > 0$ for every $(t, x) \in (T', T'') \times \Gamma_c'$ with $\Gamma_c' := \Gamma_H' \cup \Gamma_V'$. The conclusion comes as in the previous case. $\square$.

2.3 Unique continuation assumption [H4]

The goal of this section is to prove that **[H4]** holds on rectangular domains and is related to the controllability of low frequencies.

2.3.1 The case of a rectangle

Proposition 3. *Let $L > 0$, $\Omega := (0,\pi) \times (0,L)$, Γ_c be a non empty open subset of $\partial\Omega$, $R_1, R_2 \in \mathbb{N}^*$ be such that $R_1^2 + \left(\frac{R_2\pi}{L}\right)^2$ is a simple eigenvalue of $(-\Delta_D)$ and $\varphi_R(x,y) := \frac{2}{\sqrt{\pi L}} \sin(R_1 x) \sin\left(\frac{R_2 \pi y}{L}\right)$. Then **[H4]** is satisfied.*

Proof of Proposition 3: Without loss of generality, one may assume that Γ_c contains $(a,b) \times \{0\}$ for some $0 \leqslant a < b \leqslant \pi$. Let $\lambda \in \mathrm{Sp}(-\Delta_D)$, $J := \{(p,n) \in (\mathbb{N}^*)^2; p^2 + \left(\frac{n\pi}{L}\right)^2 = \lambda\}$ wich is a finite set, and $\Phi = \sum_{(p,n)\in J} c_{p,n} \sin(px) \sin\left(\frac{n\pi y}{L}\right)$ be an eigenfunction of $(-\Delta_D)$ associated to the eigenvalue λ. >From the relation $2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$ we deduce that

$$\varphi_R \Phi = \sum_{(p,n)\in J} \sum_{\substack{\varepsilon_1 = \pm 1, \\ \varepsilon_2 = \pm 1}} \frac{c_{p,n}\epsilon_1\epsilon_2}{2\sqrt{\pi L}} \cos[(p+\epsilon_1 R_1)x] \cos\left[\frac{(n+\epsilon_2 R_2)\pi y}{L}\right]$$

thus

$$w(x,y) = \sum_{(p,n)\in J} \sum_{\substack{\varepsilon_1 = \pm 1, \\ \varepsilon_2 = \pm 1}} \frac{c_{p,n}\epsilon_1\epsilon_2 \cos[(p+\epsilon_1 R_1)x] \cos\left[\frac{(n+\epsilon_2 R_2)\pi y}{L}\right]}{2\sqrt{\pi L}\left[(p+\epsilon_1 R_1)^2 + \left(\frac{(n+\epsilon_2 R_2)\pi}{L}\right)^2 + 1\right]}$$

and in particular

$$w(x,0) = \sum_{(p,n)\in J} \sum_{\substack{\varepsilon_1 = \pm 1, \\ \varepsilon_2 = \pm 1}} \frac{c_{p,n}\epsilon_1\epsilon_2 \cos[(p+\epsilon_1 R_1)x]}{2\sqrt{\pi L}\left[(p+\epsilon_1 R_1)^2 + \left(\frac{(n+\epsilon_2 R_2)\pi}{L}\right)^2 + 1\right]}.$$

Let $P := \max\{p \in \mathbb{N}^*; \exists n \in \mathbb{N}^* \text{ such that } (p,n) \in J \text{ and } c_{p,n} \neq 0\}$. Let $N \in \mathbb{N}^*$ be the unique integer such that $(P,N) \in J$. Then $x \mapsto w(x,0)$ is a finite sum of cosine functions

$$w(x,0) = \sum_{k=0}^{P+R_1} \alpha_k \cos(kx)$$

and its fastest oscilating term has coefficient

$$\alpha_{P+R_1} = \frac{c_{P,N}}{2\sqrt{\pi L}} \left(\frac{1}{\left[(P+R_1)^2 + \left(\frac{(N+R_2)\pi}{L}\right)^2 + 1\right]} - \frac{1}{\left[(P+R_1)^2 + \left(\frac{(N-R_2)\pi}{L}\right)^2 + 1\right]} \right).$$

If $w(x,0) = 0$ for every $x \in (a,b)$, then $\alpha_k = 0$ for $k = 0, ..., P+R_1$ and in particular $\alpha_{P+R_1} = 0$. This is impossible because $c_{P,N} \neq 0$ and $(N+R_2)^2 \neq (N-R_2)^2$ (note that $N, R_2 \in \mathbb{N}^*$). □

2.3.2 General comments

First, we prove the equivalence between assumption **[H4]** and the existence of some non-vanishing integral quantities. To this aim, we introduce the following notation: for $\chi \in L^2(\partial\Omega)$, V_χ denotes the unique solution in $H^{\frac{3}{2}}(\Omega)$ of

$$\begin{cases} (-\Delta + 1)V_\chi(x) = 0\,, & x \in \Omega \\ \partial_\nu V_\chi(x) = \chi(x)\,, & x \in \partial\Omega\,. \end{cases} \tag{8}$$

Proposition 4. *Let Ω be a smooth bounded open subset of $\mathbb{R}^2$ and $(\varphi_k)_{k\in\mathbb{N}}$ be an orthogonal basis of eigenfunctions of $(-\Delta_D)$, $R \in \mathbb{N}^*$. If the spectrum of $(-\Delta_D)$ is simple, then the following statements are equivalent.*

1. *For any eigenvector Φ of $(-\Delta_D)$, the solution w of*

$$\left\{\begin{array}{ll} (-\Delta+1)w(x) = \varphi_R(x)\Phi(x)\,, & x\in\Omega\,,\\ \partial_\nu w(x) = 0\,, & x\in\partial\Omega\,, \end{array}\right.$$

does not identically vanish on Γ_c: $w \neq 0$ on Γ_c.

2. *there is an open dense set in $L^2(\Gamma_c)$ of functions χ such that $\int_\Omega V_\chi \varphi_R \varphi_k \neq 0$ for any $k \in \mathbb{N}^*$*

3. *there is one $\chi \in L^2(\Gamma_c)$ such that $\int_\Omega V_\chi \varphi_R \varphi_k \neq 0$ for any $k \in \mathbb{N}^*$.*

If the spectrum of $(-\Delta_D)$ is not simple, we still have $1 \Rightarrow 2 \Rightarrow 3$.

Proof We have obvisously $2 \Rightarrow 3$. The main tool for the other implications will be the following formula

$$\begin{array}{ll} \int_\Omega V_\chi\varphi_R\Phi & = \int_\Omega V_\chi(-\Delta+1)w \\ & = \int_\Omega(-\Delta+1)V_\chi w + \int_{\partial\Omega}[\partial_\nu V_\chi w - V_\chi\partial_\nu w] \\ & = \int_{\partial\Omega}\chi w\,. \end{array}$$

In the case of simple eigenvalues, we easily get $3 \Rightarrow 1$ by selecting $\chi \in L^2(\Gamma_c)$ such that $\int_\Omega V_\chi\varphi_R\varphi_k = \int_{\partial\Omega}\chi w_k \neq 0$. So, the main task is to prove $1 \Rightarrow 2$. For $k \in \mathbb{N}^*$, we introduce the set

$$\mathcal{G}_k := \left\{\chi \in L^2(\partial\Omega,\mathbb{R});\int_\Omega V_\chi\varphi_R(x)\varphi_k(x)dx \neq 0\right\}\,,$$

By assumption 1, for any k, w_k is not identically zero on Γ_c, so there exists χ such that such $\int_{\partial\Omega}\chi w_k = \int_\Omega V_\chi\varphi_R\varphi_k \neq 0$. In particular, $\mathcal{G}_k \neq \emptyset$ for every $k \in \mathbb{N}^*$. Because of the linearity, $\mathcal{G}_k$ is a dense open subset of $L^2(\partial\Omega,\mathbb{R})$. By Baire Lemma $\cap_{k\in\mathbb{N}^*}\mathcal{G}_k$ is a dense subset of $L^2(\partial\Omega,\mathbb{R})$. This proves 2. □

With the previous Proposition in hands, it should be possible to prove that **[H4]** is generic with respect to perturbations of the domain, as done in [26]. Indeed, the previous Proposition proves that **[H4]** is equivalent to the existence of one χ so that some integral quantities are not zero. It is then often a good starting point to prove genericity with respect to χ and deformation of Ω. This would then directly imply that **[H4]** is generic with respect to perturbations of the domain.

Note also that with the same proof, χ can be chosen in $C_0^k(\Gamma_c)$ for some $k \in \mathbb{N}$.

2.3.3 Control of low frequencies

Actually, the condition **[H4]** is a condition of controllability of the low frequencies for a control only depending on time. More precisely, we have the following result.

Proposition 5. *Assume all the eigenvalues of Δ_D are simple and **[H4]** is fulfilled for one $R \in \mathbb{N}^*$. Let $\psi_{ref}(t) := \varphi_R(x)e^{-i\lambda_R t}$. Then there is one $\chi \in C^4(\Gamma_c)$ such that $\int_\Omega V_\chi \varphi_R \Phi_k \neq 0$ for any $k \in \mathbb{N}^*$. This implies the following local controllability property for low frequencies.*

For any $N \geq 0$, there exists $\delta > 0$ such that for every $\psi_f \in \mathcal{V} := \{\psi_f \in H^3_{(0)}(\Omega) \cap \mathcal{S}; \|\psi_f - \psi_{ref}(T)\|_{H^3_{(0)}} < \delta\}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that the solution of

$$\begin{cases} \Big(i\partial_t + \Delta\Big)\psi(t,x) = u(t)V_\chi(x)\psi(t,x)\,, & (t,x) \in (0,T)\times\Omega\,, \\ \psi(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \\ \psi(0,x) = \varphi_R(x) & x \in \Omega\,, \end{cases} \tag{9}$$

satisfies $\mathbb{P}_N^\perp \psi(T) = \mathbb{P}_N^\perp \psi_f$ where $\mathbb{P}_N^\perp$ is the orthogonal projection on the N first eigenfunctions.

Proof of Proposition 5: The map

$$\left| \begin{array}{rcl} \Theta: \quad L^2((0,T),\mathbb{R}) & \to & \mathbb{P}_N^\perp[\mathcal{S}] \\ u & \mapsto & \mathbb{P}_N^\perp[\psi(T)] \end{array} \right.$$

is of class C^1 and

$$\left| \begin{array}{rcl} d\Theta(0): \quad L^2((0,T),\mathbb{R}) & \to & \mathbb{P}_N^\perp[T_{\psi_{ref}(T)}\mathcal{S}] \\ u & \mapsto & \mathbb{P}_N^\perp[\Psi(T)] \end{array} \right.$$

where

$$\begin{aligned} \Psi(T) &= -i\int_0^T e^{i(T-t)\Delta_D}\Big(u(t)V_\chi\psi_{ref}(t)\Big)dt \\ &= -i\sum_{k=1}^\infty \langle V_\chi\varphi_R, \varphi_k\rangle \left(\int_0^T u(t)e^{i(\lambda_k-\lambda_R)t}dt\right) e^{-i\lambda_k T}\varphi_k \end{aligned}$$

The surjectivity of $d\Theta(0)$ can be formulated in terms of a finite trigonometric moment problem on u. Since $\langle V_\chi\varphi_R, \varphi_k\rangle \neq 0$ for every k and the frequencies $(\lambda_k - \lambda_R)$ are all different, this moment problem has a solution $u \in L^2((0,T),\mathbb{R})$. We can therefore apply the inverse function theorem to Θ. We skip the details since it is a simpler version of arguments used several times in the paper. $\square$

3 Well posedness of the Schrödinger-Poisson system and linearization

The goal of this section is to prove the well-posedness of system (1) in functional spaces that are appropriate for the controllability problem. This proof requires several preliminary results. The first one concerns Sobolev embeddings, trace theorems and elliptic estimates; they are recalled in Section 3.1. The second preliminary result concerns smoothing effects of the Schrödinger equation on a bounded domain: a smoothing effect concerning the normal derivative is recalled in Section 3.2, a smoothing effect concerning the source term is justified in Section 3.3. Finally, in Section 3.4, we prove the well-posedness of system (1) in $H^3_{(0)}(\Omega)$ when the control g lives in $L^2((0,T)\times\Gamma_c)$; the smoothing effects are crucial at this point. In Section 3.5, we prove the C^1-regularity of the end-point map.

3.1 Preliminary

3.1.1 Sobolev embeddings

In the whole section Ω is an open subset of $\mathbb{R}^n$ such that **[H1]** is satisfied. For $s \geq 0$ and $p \geq 1$, we use the definitions

$$W^{s,p}(\mathbb{R}^2) := \left\{ \widetilde{f} \in \mathcal{S}'(\mathbb{R}^2); \mathcal{F}^{-1}[(1 + |\cdot|^s)\mathcal{F}\widetilde{f}] \in L^p(\mathbb{R}^2) \right\},$$

$$\|\widetilde{f}\|_{W^{s,p}(\mathbb{R}^2)} := \left\| \mathcal{F}^{-1}[(1 + |\cdot|^s)\mathcal{F}\widetilde{f}] \right\|_{L^p(\mathbb{R}^2)},$$

$$W^{s,p}(\Omega) := \{\widetilde{f}|_\Omega; \widetilde{f} \in W^{s,p}(\mathbb{R}^2)\},$$

$$\|f\|_{W^{s,p}(\Omega)} := \inf \left\{ \left\|\widetilde{f}\right\|_{W^{s,p}(\mathbb{R}^2)}; \widetilde{f} \in W^{s,p}(\mathbb{R}^2), f = \widetilde{f} \text{ on } \Omega \right\}$$

If $p = 2$, we use the notation H^s instead of $W^{s,2}$.

Remark that if $s = 1$, we also have

$$W^{1,p}(\Omega) := \{f \in L^p(\Omega); \nabla f \in L^p(\Omega)^n\}$$

with the obvious norm equivalent to the one previously introduced.

The standard Sobolev embeddings

$$H^1(\Omega) \subset L^q(\Omega), \quad \forall q \in [2, \infty), \tag{10}$$

$$W^{1,p}(\Omega) \subset L^\infty(\Omega), \quad \forall p > 2$$

(see [12, Corollary IX.14]) and

$$W^{s,p}(\mathbb{R}^2) \subset W^{s_1,p_1}(\mathbb{R}^2) \text{ for } s - \frac{2}{p} = s_1 - \frac{2}{p_1}, 1 < p \leq p_1 < +\infty,$$

(see [7, Theorem 6.5.1]) yield to

$$H^{\frac{3}{2}}(\Omega) \subset L^\infty(\Omega), \tag{11}$$

$$H^{\frac{1}{2}}(\Omega) \subset L^4(\Omega), \tag{12}$$

$$\forall \epsilon \in \left(0, \frac{1}{2}\right), \exists p \in (1, 2) \text{ such that } W^{1,p}(\Omega) \subset H^{\frac{1}{2}+\epsilon}(\Omega). \tag{13}$$

If $s \geq 0$, we denote

$$H^{-s}(\Omega) := (H_0^s(\Omega))'$$

which is a space of distributions.

3.1.2 Regularity for elliptic boundary value problems on a smooth domain

In this section, we recall classical results used in the article (see [22, Theorem 5.4 page 176, Theorem 6.5, Theorem 6.7 page 192]).

Proposition 6. *Let Ω be a bounded open subset of $\mathbb{R}^2$, of class C^∞ and locally on one side of $\partial\Omega$.*

1. *The mapping*
$$\left| \begin{array}{ccc} C^\infty(\overline{\Omega}) & \to & C^\infty(\partial\Omega) \\ f & \mapsto & f|_{\partial\Omega} \end{array} \right.$$
has a unique continuous extension from $D(\Delta, L^2(\Omega)) := \{f \in L^2(\Omega); \Delta f \in L^2(\Omega)\}$ into $H^{-\frac{1}{2}}(\partial\Omega)$.

2. *The mapping*
$$\left| \begin{array}{ccc} H^2(\Omega) & \to & L^2(\partial\Omega) \\ f & \mapsto & \partial_\nu f \end{array} \right.$$
has a unique continuous extension from $\{f \in H^{\frac{3}{2}}(\Omega); \Delta f \in L^2(\Omega)\}$ into $L^2(\partial\Omega)$.

3. *For every $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, there exists a unique $v \in H^{\frac{3}{2}}(\Omega)$ such that*
$$\begin{cases} (-\Delta + 1)v = f \text{ in } \Omega\,, \\ \partial_\nu v = g \text{ on } \partial\Omega\,. \end{cases}$$
Moreover, there exists $C(\Omega) > 0$ such that
$$\|v\|_{H^{\frac{3}{2}}(\Omega)} \leqslant C\Big[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\Big]\,.$$

3.1.3 Regularity and Green formula for elliptic boundary value problems on a rectangle

The goal of this section is to state the following results.

Proposition 7. *Let $\Omega = (0,\pi) \times (0,L)$ for some $L > 0$, $(\Gamma_j)_{1\leqslant j\leqslant 4}$ be its edges, $(\nu_j)_{1\leqslant j\leqslant 4}$ be its unitary exterior normal vectors, $(S_j)_{1\leqslant j\leqslant 4}$ be its vertices and $p \geqslant 1$.*

1. *The mapping $u \mapsto (u|_{\Gamma_j}, \partial_{\nu_j} u)$, which is well defined for $u \in W^{2,p}(\Omega)$ has a unique extension as an operator from*
$$D(\Delta, L^p(\Omega)) := \{u \in L^p(\Omega); \Delta u \in L^p(\Omega)\}$$
into
   - *$W^{-\frac{1}{p},p}(\Gamma_j) \times W^{-1-\frac{1}{p},p}(\Gamma_j)$ when $p \neq 2$,*
   - *$H^{-\frac{1}{2}-\epsilon}(\Gamma_j) \times H^{-\frac{3}{2}-\epsilon}(\Gamma_j)$, for every $\epsilon > 0$, when $p = 2$.*

2. *Moreover,*
$$\int_\Omega \Big((\Delta u)v - u(\Delta v)\Big)dx = \sum_{j=1}^{4} \int_{\Gamma_j} \Big((\partial_{\nu_j} u)v - u(\partial_{\nu_j} v)\Big)d\sigma_j(x)$$
for every $u \in D(\Delta, L^p(\Omega))$ and $v \in W^{2,p'}(\Omega)$ such that

- $v(S_j) = 0$ *for* $j = 1, ..., 4$ *when* $p > 2$,
- $v(S_j) = 0$ *and* $\nabla v(S_j) = 0$ *for* $j = 1, ..., 4$ *when* $p < 2$,
- $v \equiv 0$ *on a neighborhood of* S_j *for* $j = 1, ..., 4$ *when* $p = 2$.

3. *For every* $f \in L^2(\Omega)$ *and* $g \in L^2(\partial\Omega)$, *there exists a unique* $v \in H^{\frac{3}{2}}(\Omega)$ *such that*

$$\begin{cases} (-\Delta + 1)v = f \text{ in } \Omega\,, \\ \partial_\nu v = g \text{ on } \partial\Omega\,. \end{cases} \tag{14}$$

Moreover, there exists $C(\Omega) > 0$ *such that*

$$\|v\|_{H^{\frac{3}{2}}(\Omega)} \leqslant C\Big[\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\Big]\,.$$

For the first two statements, see [18, Theorem 1.5.3.4 and Theorem 1.5.3.6]. The third one is a consequenc of the same one on $(\mathbb{R}/2\pi\mathbb{Z}) \times (0, L)$ after symmetrisation and periodization. More precisely, by linearity, one may assume that $\mathrm{Supp}(g) \subset [0, \pi] \times \{0\}$. Then we extend g as a function $\widetilde{g} : (\mathbb{R}/\pi\mathbb{Z}) \times \{0\} \to \mathbb{R}$ such that $\widetilde{g}(-x_1, 0) = g(x_1, 0)$ for every $x_1 \in (0, \pi)$ and $\widetilde{g}(x_1, 0) = \widetilde{g}(x_1 + 2\pi, 0)$. Perform similar symmetrisation and periodization for f to produce $\widetilde{g}$ well defined on $(\mathbb{R}/2\pi\mathbb{Z}) \times (0, L)$.

Then, since $(\mathbb{R}/2\pi\mathbb{Z}) \times (0, L)$ is a smooth compact manifold with boundary, there exists a unique $\widetilde{v} \in H^{\frac{3}{2}}((\mathbb{R}/2\pi\mathbb{Z}) \times (0, L))$ such that

$$\begin{cases} (-\Delta + 1)\widetilde{v} = \widetilde{f} \text{ in } \Omega\,, \\ \partial_\nu \widetilde{v} = \widetilde{g} \text{ on } (\mathbb{R}/\pi\mathbb{Z}) \times \{0, 1\}\,. \end{cases}$$

Then, we easily check that the symmetry gives that if $f \in C_0^\infty(\Omega)$ and $g \in C_0^\infty(0, \pi)$, then, $\partial_{x_1}\widetilde{v}(x_1, x_2) = 0$ for $x_1 \in \{0, \pi\}$ and $x_2 \in (0, L)$. Therefore, $v = \widetilde{v}_{|\Omega}$ satisfies (14) in the weak sense. The uniqueness can be obtained by following the same process.

3.2 Smoothing effect on the normal derivative

The goal of this section is to state the following results.

Proposition 8. *Let* $T > 0$ *and* Ω *be a bounded open subset of* $\mathbb{R}^2$ *which is either* C^∞ *or a rectangle. There exists* $C = C(T, \Omega) > 0$ *such that, for every* $\psi_0 \in H_0^1(\Omega)$ *and* $h \in L^1((0, T), H_0^1(\Omega))$, *the solution* ψ *of*

$$\begin{cases} (i\partial_t + \Delta)\psi(t, x) = h(t, x)\,, & (t, x) \in (0, T) \times \Omega\,, \\ \psi(t, x) = 0\,, & (t, x) \in (0, T) \times \partial\Omega\,, \\ \psi(0, x) = \psi_0(x)\,, & x \in \Omega\,, \end{cases} \tag{15}$$

satisfies

$$\int_0^T \int_{\partial\Omega} |\partial_\nu \psi(t, x)|^2 |\partial_\nu \varphi_1(x)| d\sigma(x) dt \leqslant C\Big(\|\psi_0\|_{H_0^1(\Omega)} + \|h\|_{L^1((0,T),H_0^1(\Omega))}\Big)\,. \tag{16}$$

In particular, if Γ_c *is an open subset of* $\partial\Omega$ *such that* ***[H1]*** *holds, then the following linear mapping is continuous*

$$\left| \begin{array}{ccccc} H_0^1(\Omega) & \times & L^1((0, T), H_0^1(\Omega)) & \to & L^2((0, T) \times \Gamma_c) \\ (\psi_0 & , & \varphi) & \mapsto & \partial_\nu \psi\,. \end{array} \right.$$

When Ω is smooth, Puel proves these results in [34, Lemma 3.1]. Precisely, he first proves inequality (16) for smooth data $(\psi_0, h) \in C_c^\infty(\Omega) \times C_c^\infty((0,T)\times\Omega)$, by applying the multiplier $\nabla\varphi_1$ to equation (15): under these assumptions ψ is regular, $\frac{\partial\psi}{\partial\nu}$ makes perfect sense and integrations by parts are legitimate. Then, inequality (16) holds for $(\psi_0, h) \in H_0^1(\Omega) \times L^1((0,T), H_0^1(\Omega))$ by a density argument. Finally, Puel gets the final result because $|\partial_\nu \varphi_1(x)| \geqslant \beta > 0\,, \forall x \in \partial\Omega$

When Ω is a rectangle, Puel's proof of inequality (16) is still valid. Then, the conclusion of Proposition 8 follows because $|\partial_\nu \varphi_1(x)| \geqslant \beta > 0\,, \forall x \in \Gamma_c$; this is one of the reasons why we need to assume that Γ_c does not touch the vertices of Ω.

3.3 Smoothing effect on the source term

The goal of this section is to justify the following result.

Theorem 5. *Let $T>0$, Ω be an open subset of $\mathbb{R}^2$ and Γ_c be an open subset of $\partial\Omega$ such that **[H1]** holds. For every $\psi_0 \in H^3_{(0)}(\Omega)$, $\mu_1 \in L^2((0,T), H^2 \cap H^1_0(\Omega))$ and $\mu_2 \in L^1((0,T), H^3_{(0)}(\Omega))$ such that*

$$\Delta^2 \mu_1 \equiv 0\,, \quad \mathit{Supp}\left(\Delta\mu_1\Big|_{\partial\Omega}\right) \subset [0,T] \times \Gamma_c\,, \quad \Delta\mu_1\Big|_{\Gamma_c} \in L^2((0,T)\times\Gamma_c)\,, \tag{17}$$

the solution of

$$\begin{cases} \Big(i\partial_t + \Delta\Big)\psi(t,x) = \Big(\mu_1 + \mu_2\Big)(t,x)\,, & (t,x) \in (0,T)\times\Omega\,, \\ \psi(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \\ \psi(0,x) = \psi_0(x)\,, & x \in \Omega\,, \end{cases}$$

satisfies $\psi \in C^0([0,T], H^3_{(0)}(\Omega))$. Moreover, there exists a constant $C>0$ (independent of ψ_0, μ_1, μ_2) such that

$$\|\psi\|_{L^\infty([0,T],H^3_{(0)}(\Omega))} \leqslant C\Big(\|\psi_0\|_{H^3_{(0)}(\Omega)} + \left\|\Delta\mu_1\Big|_{\Gamma_c}\right\|_{L^2((0,T)\times\Gamma_c)} + \|\mu_2\|_{L^1((0,T),H^3_{(0)}(\Omega))}\Big)\,. \tag{18}$$

Remark 1. *When Ω is smooth, then the trace $\Delta\mu_1|_{\partial\Omega}$ is well defined in $L^2((0,T), H^{-\frac{1}{2}}(\partial\Omega))$ (see Proposition 6), which gives a sense to the last 2 requirements in (17).*

When Ω is a rectangle, then the traces along the 4 sides $\Delta\mu_1|_{\Gamma_j}$, $1 \leqslant j \leqslant 4$, are well defined in $L^2((0,T), H^{-\frac{1}{2}-\epsilon}(\Gamma_j))$ (see Proposition 7). The last 2 requirements in (17) have to be interpreted in the following sense:

$$\Gamma_j \cap \mathit{Supp}\left(\Delta\mu_1\Big|_{\partial\Omega}\right) \subset \Gamma_j \cap \Gamma_c\,, \quad \forall j \in \{1,...,4\}$$

$$\Delta\mu_1\Big|_{\Gamma_c\cap\Gamma_j} \in L^2((0,T)\times\Gamma_c\cap\Gamma_j)\,, \quad \forall j \in \{1,...,4\}\,.$$

Theorem 5 emphasizes a regularizing effect, concerning the source term, because μ_1 is not assumed to belong to $L^1((0,T), H^3_{(0)}(\Omega))$. When Ω is a regular domain and $\Gamma_c = \partial\Omega$, Puel proves this result in [34, Theorem 2.1]. His proof, by transposition, relies on the smoothing effect of Proposition 8.

Puel does not treat the case of a rectangle domain, but his proof would probably lead to Theorem 5 in this case too. For sake of completeness, we propose below an alternative argument.

Proof of Theorem 5 on a rectangle: Let $L > 0$, $\Omega := (0,\pi) \times (0,L)$, Γ_c be an open subset of $\partial\Omega$, $\psi_0 \in H^3_{(0)}(\Omega)$, $\mu_1 \in L^2((0,T), H^2 \cap H^1_0(\Omega))$ and $\mu_2 \in L^1((0,T), H^3_{(0)}(\Omega))$ be such that (17) holds. By linearity, we may assume that $\Gamma_c \subset (a,b)\times\{0\}$ with $0 < a < b < \pi$. Let $m \in \mathbb{N}^*$ be such that $(m-1)\frac{2L^2}{\pi} < T \leqslant T_0 := \frac{2mL^2}{\pi}$ and $t \in (0,T)$.

By the Duhamel formula, we have

$$\psi(t) = e^{i\Delta_D t}\psi_0 - i\int_0^t e^{i\Delta_D (t-\tau)}\Big(\mu_1(\tau) + \mu_2(\tau)\Big)d\tau\,.$$

The first and third terms in the right hand side belong to $C^0([0,T], H^3_{(0)}(\Omega))$ because $H^3_{(0)}(\Omega)$ is stable by $e^{i\Delta_D t}$. Thus, we focus on the second one. We will make the proof in 2 steps: the first one when μ_1 is regular enough to perform integration by parts and the second one to get to the first case with a suitable regularization.

Step 1: μ_1 regular enough

In this step, we assume moreover that $\Delta\mu_1 \in L^2((0,T), L^p(\Omega))$, with $p > 2$.

Using the equality $-\Delta\varphi_k = \lambda_k\varphi_k$ and integrations by part, that are licit because $\mu_1 \in L^2((0,T), H^2 \cap H^1_0(\Omega))$, we get

$$\begin{aligned} S = \left\|\int_0^t e^{i\Delta(t-\tau)}\mu_1(\tau)d\tau\right\|^2_{H^3_{(0)}(\Omega)} &= \sum_{k=1}^{\infty}\left|\lambda_k^{3/2}\int_0^t\int_\Omega \mu_1(\tau,x)\varphi_k(x)dx e^{i\lambda_k\tau}d\tau\right|^2 \\ &= \sum_{k=1}^{\infty}\left|\frac{1}{\sqrt{\lambda_k}}\int_0^t\int_\Omega \Delta\mu_1(\tau,x)\Delta\varphi_k(x)dx e^{i\lambda_k\tau}d\tau\right|^2 \end{aligned}$$

We can apply Green formula (see Proposition 7) because $\Delta\mu_1(\tau,.) \in D(\Delta, L^p(\Omega))$ for almost every $\tau \in (0,T)$ and $\varphi_k \in H^2(\Omega)$ vanishes on the vertices of Ω. Using $\Delta^2\mu_1 \equiv 0$, this leads to

$$\begin{aligned} S &= \sum_{k=1}^{\infty}\left|\frac{1}{\sqrt{\lambda_k}}\int_0^t\int_{\partial\Omega} \Delta\mu_1(\tau,x)\partial_\nu\varphi_k(x)d\sigma(x)e^{i\lambda_k\tau}d\tau\right|^2 \\ &= \sum_{k=1}^{\infty}\left|\frac{1}{\sqrt{\lambda_k}}\int_0^t\int_{\Gamma_c} \Delta\mu_1(\tau,x)\partial_y\varphi_k(x)d\sigma(x)e^{i\lambda_k\tau}d\tau\right|^2 \\ &\leqslant C\sum_{p,n\in\mathbb{N}^*}\left|\frac{\frac{n\pi}{L}}{\sqrt{p^2+\left(\frac{n\pi}{L}\right)^2}}\int_0^t\int_0^\pi \Delta\mu_1(\tau,x_1,0)\sin(px_1)e^{-i\left(p^2+\left(\frac{n\pi}{L}\right)^2\right)\tau}dx_1d\tau\right|^2 \\ &\leqslant C\sum_{p\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*}\left|\int_0^{T_0} 1_{[0,t]}(\tau)\int_0^\pi \Delta\mu_1(\tau,x_1,0)\sin(px_1)dx_1 e^{-i\left(p^2+\left(\frac{n\pi}{L}\right)^2\right)\tau}d\tau\right|^2 \\ &\leqslant C\sum_{p\in\mathbb{N}^*}\int_0^{T_0}\left|1_{[0,t]}(\tau)\int_0^\pi \Delta\mu_1(\tau,x_1,0)\sin(px_1)dx_1 e^{-ip^2\tau}\right|^2 d\tau \\ &\leqslant C\sum_{p\in\mathbb{N}^*}\int_0^t\left|\int_0^\pi \Delta\mu_1(\tau,x_1,0)\sin(px_1)dx_1\right|^2 d\tau \\ &\leqslant C\int_0^t \|\Delta\mu_1(\tau,.,0)\|^2_{L^2(0,\pi)}d\tau\,. \end{aligned}$$

To go from the first line to the second one, we have used: $\mathrm{Supp}(\Delta\mu_1|_{\partial\Omega}) \subset [0,T]\times\Gamma_c$, $\varphi_k \equiv 0$ on Γ_c. To go from the fourth line to the fifth line, we have used Bessel-Parseval inequality.

Note that $\Delta\mu_1$ can be computed explicitly in terms of $\Delta\mu_1|_{\Gamma_c}$, thanks to the rectangular form of the domain. This explicit expression shows the existence of a constant $C = C(\Omega) > 0$ such that

$$\|\Delta\mu_1\|_{L^2((0,t)\times\Omega))} \leqslant C \left\| \Delta\mu_1\Big|_{\Gamma_c} \right\|^2_{L^2((0,t)\times\Gamma_c)}, \quad \forall t\in[0,T].$$

Thus, we get

$$\left\| \int_0^t e^{i\Delta(t-\tau)}\mu_1(\tau)d\tau \right\|^2_{H^3_{(0)}(\Omega)} \leqslant C\,\|\Delta\mu_1|_{\Gamma_c}\|^2_{L^2((0,t)\times\Gamma_c)}.$$

This proves that the map $t\in[0,T]\mapsto \int_0^t e^{i\Delta(t-\tau)}\mu_1(\tau)d\tau$ takes values in $H^3_{(0)}(\Omega)$ and is continuous at $t=0$. The same argument proves the continuity at any $t\in[0,T]$.

Step 2: regularization

We have to be a bit careful with the regularization to keep the required conditions. Let $f_\varepsilon \in C_0^\infty((0,T)\times\Gamma_c)$ converging strongly to $\Delta\mu_1|_{\Gamma_c}$ in $L^2((0,T)\times\Gamma_c)$. Denote

$$\left\{ \begin{array}{ll} \Delta g_\varepsilon(t,x) = 0\,, & (t,x)\in(0,T)\times\Omega\,, \\ g_\varepsilon(t,x) = f_\varepsilon(t,x)\,, & (t,x)\in(0,T)\times\partial\Omega\,, \end{array}\right.$$

By elliptic regularity (similar to Proposition 7), g_ε converges strongly to $\Delta\mu_1$ in $L^2((0,T),H^{1/2}(\Omega))$.

Now, denote $\mu_{1,\varepsilon}$ the solution of

$$\left\{ \begin{array}{ll} \Delta\mu_{1,\varepsilon}(t,x) = g_\varepsilon(t,x)\,, & (t,x)\in(0,T)\times\Omega\,, \\ \mu_{1,\varepsilon}(t,x) = 0\,, & (t,x)\in(0,T)\times\partial\Omega. \end{array}\right.$$

By elliptic regularity, $\mu_{1,\varepsilon}$ converges to μ_1 in $L^2((0,T),H^2\cap H^1_0(\Omega))$. Estimate (18) (that is proven for $\mu_{1,\varepsilon}$ regular enough thanks to Step 1) allows to prove that the related ψ_ε make a Cauchy sequence in $L^\infty([0,T],H^3_{(0)}(\Omega))$. The limit can only be the expected ψ by uniqueness of the solution and convergence of $\mu_{1,\varepsilon}$ to μ_1. The estimate (18) follows in this case. □

3.4 Well posedness of the linear PDE

The goal of this section is the proof of the following result, thanks to Theorem 5.

Theorem 6. *Let $T>0$, Ω be an open subset of $\mathbb{R}^2$ and Γ_c be an open subset of $\partial\Omega$ such that **[H1]** holds. For every $\psi_0\in H^3_{(0)}(\Omega)$ and $g\in L^2((0,T)\times\partial\Omega,\mathbb{R})$, there exists a unique solution $\psi\in C^0([0,T],H^3_{(0)}(\Omega))$ of system (1).*

To this aim, we will need the following preliminary result.

Proposition 9. *Let $T>0$ and $\psi\in C^0([0,T],H^3_{(0)}(\Omega))$. For every $g\in L^2((0,T)\times\partial\Omega,\mathbb{R})$, the solution of*

$$\left\{ \begin{array}{ll} (-\Delta+1)v(t,x) = 0\,, & (t,x)\in(0,T)\times\Omega\,, \\ \partial_\nu v(t,x) = g(t,x)1_{\Gamma_c}(x)\,, & (t,x)\in(0,T)\times\partial\Omega\,, \end{array}\right. \tag{19}$$

satisfies $v \in L^2((0,T), H^{\frac{3}{2}}(\Omega))$, $v\psi \in L^2((0,T), H^2 \cap H^1_0(\Omega))$ *and* $\Delta^2[v\psi]$ *belongs to* $L^2((0,T), H^{-1}(\Omega))$. *Moreover, the operator*

$$\left| \begin{array}{rcl} L^2((0,T)\times\partial\Omega,\mathbb{R}) & \to & L^2((0,T),H^{-1}(\Omega)) \\ g & \mapsto & \Delta^2(v\psi) \end{array} \right.$$

is continuous.

Proof of Proposition 9:
Step 1: We prove that $v\psi \in L^2((0,T), H^2 \cap H^1_0(\Omega))$. By Proposition 6 and 7, we know that $v \in L^2((0,T), H^{\frac{3}{2}}(\Omega))$. Thus $v\psi \in L^2((0,T), H^{\frac{3}{2}}(\Omega) \cap H^1_0(\Omega))$ because $H^{\frac{3}{2}}(\Omega) \subset L^\infty$ is an algebra. To end Step 1, it is sufficient to prove that $\Delta(v\psi) \in L^2((0,T), L^2(\Omega))$. Using $\Delta v = v$ in $(0,T)\times\Omega$, we get

$$\Delta(v\psi) = v(\psi + \Delta\psi) + 2\nabla v \cdot \nabla\psi$$

- The Sobolev embedding (11) justifies that $H^{\frac{3}{2}}(\Omega) * H^1_0(\Omega) \subset L^\infty(\Omega) * L^2(\Omega) = L^2(\Omega)$ (where $*$ stands for the multiplication of scalar valued functions). Thus $v(\psi+\Delta\psi) \in L^2((0,T), L^2(\Omega))$.

- The Sobolev embedding (11) justifies that $H^{\frac{1}{2}}(\Omega) * H^2(\Omega) \subset L^2(\Omega) * L^\infty(\Omega) = L^2(\Omega)$. Thus $\nabla v \cdot \nabla\psi \in L^2((0,T), L^2(\Omega))$.

Step 2: We prove that $\Delta^2[v\psi] \in L^2((0,T), H^{-1}(\Omega))$. Using $\Delta v = v$, we get

$$\Delta^2[v\psi] = v(\psi + 2\Delta\psi + \Delta^2\psi) + 4\nabla v \cdot \nabla(\psi + \Delta\psi) + 4\mathrm{Tr}[D^2 v\,.D^2\psi]\,. \tag{20}$$

- The Sobolev embeddings (11) and (12) justify that $H^{\frac{3}{2}}(\Omega) * H^1_0(\Omega) \subset H^1_0(\Omega)$. Thus, by duality, $H^{\frac{3}{2}}(\Omega) * H^{-1}(\Omega) \subset H^{-1}(\Omega)$. This proves that $v(\psi + 2\Delta\psi + \Delta^2\psi) \in L^2((0,T), H^{-1}(\Omega))$.

- The Sobolev embedding (12) justify that $H^{\frac{1}{2}}(\Omega) * H^1_0(\Omega) \subset L^4(\Omega) * L^4(\Omega) \subset L^2(\Omega)$. Thus, by duality, $H^{\frac{1}{2}}(\Omega) * L^2(\Omega) \subset H^{-1}(\Omega)$. This proves that $\nabla v \cdot \nabla(\psi + \Delta\psi) \in L^2((0,T), H^{-1}(\Omega))$.

- The Sobolev embedding (10) and then (13) justify that $H^1(\Omega) * H^1_0(\Omega) \subset H^{\frac{1}{2}+\epsilon}(\Omega)$. Thus, by duality, $H^1(\Omega) * H^{-\frac{1}{2}-\epsilon}(\Omega) \subset H^{-1}(\Omega)$. This proves that $\mathrm{Tr}[D^2 v\,.D^2\psi] \in L^2((0,T), H^{-1}(\Omega))$. □

Proof of Theorem 6: Let $T>0$, $\psi_0 \in H^3_{(0)}(\Omega)$, $g \in L^2((0,T)\times\partial\Omega,\mathbb{R})$ and v be the solution of (19) We apply the fixed point theorem to the map

$$\left| \begin{array}{rcl} F: \quad C^0([0,T], H^3_{(0)}(\Omega)) & \to & C^0([0,T], H^3_{(0)}(\Omega)) \\ \psi & \mapsto & \xi \end{array} \right.$$

where $\xi = F(\psi)$ is the solution of

$$\left\{ \begin{array}{ll} \Big(i\partial_t + \Delta\Big)\xi(t,x) = v(t,x)\psi(t,x)\,, & (t,x) \in (0,T)\times\Omega\,, \\ \xi(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \\ \xi(0,x) = \psi_0(x)\,, & x \in \Omega\,. \end{array} \right.$$

Step 1: We prove that F takes values in $C^0([0,T],H^3_{(0)}(\Omega))$. Let $\psi \in C^0([0,T],H^3_{(0)}(\Omega))$. We introduce the solution μ_2 of

$$\left\{\begin{array}{ll} \Delta^2\mu_2(t,x)=\Delta^2[v\psi](t,x)\,, & (t,x)\in(0,T)\times\Omega\,,\\ \mu_2(t,x)=\Delta\mu_2(t,x)=0\,, & (t,x)\in(0,T)\times\partial\Omega \end{array}\right. \tag{21}$$

and the function $\mu_1 := v\psi - \mu_2$. To prove that $\xi = F(\psi)$ belongs to $C^0([0,T],H^3_{(0)}(\Omega))$, it suffices to prove that μ_1 and μ_2 satisfy the assumptions of Theorem 5.

By Proposition 9, $\Delta^2[v\psi]$ belongs to $L^2((0,T),H^{-1}(\Omega))$ thus, by elliptic regularity, $\mu_2 \in L^2((0,T),H^3_{(0)}(\Omega))$. Then, μ_1 belongs to $L^2((0,T),H^2\cap H^1_0(\Omega))$, by Proposition 9. Moreover, $\Delta^2\mu_1\equiv 0$ by (21) and $\Delta\mu_1|_{\partial\Omega} = 2g\,\partial_\nu\psi\,1_{\Gamma_c}$.

Step 2: We prove that F is a contraction when g is small enough. Let $\psi,\widetilde{\psi} \in C^0([0,T],H^3_{(0)}(\Omega))$. >From (18) and elliptic regularity, we get

$$\begin{aligned} &\|F(\psi)-F(\widetilde{\psi})\|_{C^0([0,T],H^3_{(0)}(\Omega))}\\ \leqslant\ & C\Big(\|\Delta(\mu_1-\widetilde{\mu}_1)\|_{L^2((0,T)\times\Gamma_c)} + \|\mu_2-\widetilde{\mu}_2\|_{L^1((0,T),H^3_{(0)}(\Omega))}\Big)\\ \leqslant\ & C'\|g\|_{L^2((0,T)\times\Gamma_c)}\|\psi-\widetilde{\psi}\|_{C^0([0,T],H^3_{(0)}(\Omega))}\,. \end{aligned}$$

for some constant C' independant of ψ and $\widetilde{\psi}$. Thus F is a contraction when $C'\|g\|_{L^2((0,T)\times\Gamma_c)} < 1$. Otherwise, one may subdivide the interval $(0,T)$ into a finite number of intervals on which this assumption is satisfied and iterate the previous result. □

3.5 C^1-regularity of the end point map

Proposition 10. *Let $T>0$, Ω be an open subset of $\mathbb{R}^2$ and Γ_c be an open subset of $\partial\Omega$ such that **[H1]** holds. Consider the end point map*

$$\left|\begin{array}{lllll} \Theta: & [H^3_{(0)}(\Omega)\cap\mathcal{S}] & \times\ L^2((0,T)\times\Gamma_c,\mathbb{R}) & \to & H^3_{(0)}(\Omega)\cap\mathcal{S}\\ & (\psi_0 & ,\ \ g) & \mapsto & \psi(T) \end{array}\right.$$

where ψ is the solution of (1). Then, Θ is C^1 and for every $\psi_0\in[H^3_{(0)}(\Omega)\cap\mathcal{S}]$,

$$\left|\begin{array}{lllll} d\Theta(\psi_0,0): & [H^3_{(0)}(\Omega)\cap T_{\mathcal{S}}\psi_0] & \times\ L^2((0,T)\times\Gamma_c,\mathbb{R}) & \to & H^3_{(0)}(\Omega)\cap T_{\mathcal{S}}\psi_{ref}(T)\\ & (\Psi_0 & ,\ \ G) & \mapsto & \Psi(T) \end{array}\right.$$

where $\psi_{ref}(t) := e^{i\Delta_D t}\psi_0$ and Ψ is the solution of the linearized system

$$\left\{\begin{array}{ll} \Big(i\partial_t+\Delta\Big)\Psi(t,x)=V(t,x)\psi_{ref}(t,x)\,, & (t,x)\in(0,T)\times\Omega\,,\\ \Psi(t,x)=0\,, & (t,x)\in(0,T)\times\partial\Omega\,,\\ \Psi(0,x)=\Psi_0(x)\,, & x\in\Omega\,,\\ (-\Delta+1)V(t,x)=0\,, & (t,x)\in(0,T)\times\Omega\,,\\ \partial_\nu V(t,x)=G(t,x)1_{\Gamma_c}(x)\,, & (t,x)\in(0,T)\times\partial\Omega\,. \end{array}\right. \tag{22}$$

This result is a consequence of estimate (18). Its proof is classical and follows the same steps as in [5, Proof of Proposition 3, Pages 531-532].

4 Local exact control of high frequencies

The goal of this section is the proof of Theorem 1, via a perturbation argument. In Section 4.1, we prove the controllability of the linearized system at high frequency, thanks to the weak observability inequality of the adjoint system and the Hilbert Uniqueness Method. In Section 4.2, we prove Theorem 1 by applying the inverse mapping theorem.

4.1 Control of high frequencies for the linearized system

The goal of this section is to prove the following result.

Proposition 11. *Let $T > 0$, $\psi_0 \in H^3_{(0)}(\Omega) \cap \mathcal{S}$ and $\psi_{ref}(t) := e^{i\Delta_D t}\psi_0$. We assume that **[H1]**, **[H2]** and **[H3]** hold. Then, there exists $K \in \mathbb{N}^*$ and a continuous linear map*

$$L : \mathbb{P}_K[H^3_{(0)}(\Omega)] \to L^2((0,T) \times \Gamma_c, \mathbb{R})$$

such that, for every $\Psi_f \in \mathbb{P}_K[H^3_{(0)}(\Omega)]$, the solution of (22) with $\Psi_0 = 0$ and control $G = L[\Psi_f]$ satisfies $\mathbb{P}_K[\Psi(T)] = \Psi_f$.

The following proposition will allow to work on $r := \Delta^2\Psi$ instead of Ψ.

Proposition 12. *Let $T > 0$, $\psi_0 \in H^3_{(0)}(\Omega) \cap \mathcal{S}$, $\psi_{ref}(t) := e^{i\Delta_D t}\psi_0$, $G \in L^2((0,T) \times \Gamma_c, \mathbb{R})$ and $\Psi \in C^0([0,T], H^3_{(0)}(\Omega))$ be the solution of system (22) with $\Psi_0 = 0$. Then, the function $r := \Delta^2[\Psi]$ belongs to $C^0([0,T], H^{-1}(\Omega))$ and is the solution by transposition of*

$$\begin{cases} \Big(i\partial_t + \Delta\Big) r(t,x) = \mathcal{K}[G](t,x)\,, & (t,x) \in (0,T) \times \Omega\,, \\ r(t,x) = 2G\partial_\nu \psi_{ref} 1_{\Gamma_c}\,, & (t,x) \in (0,T) \times \partial\Omega\,, \\ r(0,x) = 0\,, & x \in \Omega\,, \end{cases} \tag{23}$$

where

$$\left| \begin{array}{rcl} \mathcal{K} : L^2((0,T) \times \Gamma_c, \mathbb{R}) & \to & L^2((0,T), H^{-1}(\Omega)) \\ G & \mapsto & \Delta^2[V\psi_{ref}] \text{ where } \begin{cases} (-\Delta + 1)V = 0 \text{ in } (0,T) \times \Omega\,, \\ \partial_\nu V = G 1_{\Gamma_c} \text{ on } (0,T) \times \partial\Omega\,. \end{cases} \end{array} \right. \tag{24}$$

This means that, for every $\phi_T \in H^1_0(\Omega)$ and $\varphi \in L^1((0,T), H^1_0(\Omega))$ the following equality holds

$$\begin{aligned} \int_0^T \langle r(t), \overline{\varphi(t)} \rangle_{H^{-1}, H^1_0} dt = & \int_0^T \langle \mathcal{K}(G)(t), \overline{\phi(t)} \rangle_{H^{-1}, H^1_0} dt - i \langle r(T), \overline{\phi_T} \rangle_{H^{-1}, H^1_0} \\ & + 2 \int_0^T \int_{\Gamma_c} G(t,x) \partial_\nu \psi_{ref}(t,x) \overline{\partial_\nu \phi(t,x)} d\sigma(x) dt \end{aligned} \tag{25}$$

where $\phi \in C^0([0,T], H^1_0(\Omega))$ is the solution of

$$\begin{cases} \Big(i\partial_t + \Delta\Big) \phi(t,x) = \varphi(t,x)\,, & (t,x) \in (0,T) \times \Omega\,, \\ \phi(t,x) = 0\,, & (t,x) \in (0,T) \times \partial\Omega \\ \phi(T,x) = \phi_T(x)\,, & x \in \Omega\,. \end{cases}$$

Note that, there is no ambiguity in this definition because for every $\phi_T \in H_0^1(\Omega)$ and $\varphi \in L^1((0,T), H_0^1(\Omega))$ then $\partial_\nu \phi$ belongs to $L^2((0,T)\times\Gamma_c)$ by Proposition 8.

Proof of Proposition 12: When $G \in C_c^\infty((0,T)\times\Gamma_c)$, $\varphi \in C_c^\infty((0,T)\times\Omega)$ and $\phi_T \in C_c^\infty(\Omega)$, then, formula (25) can be proved with integrations by part. The conclusion follows thanks to the continuity of the maps $\mathcal{K}$ (see Proposition 9) and the continuity of the following linear mappings

$$\begin{array}{ccccc} L^2((0,T)\times\Gamma_c) & \to & C^0([0,T], H^3_{(0)}(\Omega)) & \to & C^0([0,T], H^{-1}(\Omega)) \\ G & \mapsto & \Psi & \mapsto & r = \Delta^2\Psi \end{array}$$

$$\begin{array}{ccc} L^1((0,T), H_0^1(\Omega)) \times H_0^1(\Omega) & \to & L^2((0,T)\times\Gamma_c) \\ (\varphi, \phi_T) & \mapsto & \partial_\nu \phi \end{array}$$

stated in Theorem 5 and Proposition 8. □

From now on, we denote by $\mathcal{K}^*$ the adjoint operator of $\mathcal{K}$,

$$\mathcal{K}^* : L^2((0,T), H_0^1(\Omega)) \to L^2((0,T)\times\Gamma_c, \mathbb{R}),$$

i.e.

$$\Re\left(\int_0^T \langle \mathcal{K}(G), \overline{\xi}\rangle_{H^{-1},H_0^1} dt\right) = \int_0^T \int_{\Gamma_c} G\mathcal{K}^*(\xi) d\sigma(x) dt \tag{26}$$

for every $\xi \in L^2((0,T), H_0^1(\Omega))$ and $G \in L^2((0,T)\times\Gamma_c, \mathbb{R})$.

Thus Proposition 11 is a consequence of the following result.

Proposition 13. *Let $T > 0$, $\psi_0 \in H^3_{(0)}(\Omega) \cap \mathcal{S}$ and $\psi_{ref}(t) := e^{i\Delta_D t}\psi_0$. We assume that **[H1]**, **[H2]** and **[H3]** hold. Then, there exists $K \in \mathbb{N}^*$ and a continuous linear map*

$$\widetilde{L} : \mathbb{P}_K[H^{-1}_{(0)}(\Omega)] \to L^2((0,T)\times\Gamma_c, \mathbb{R})$$

such that, for every $r_f \in \mathbb{P}_K[H^{-1}_{(0)}(\Omega)]$, the solution of (23) with control $G = \widetilde{L}[r_f]$ satisfies $\mathbb{P}_K[r(T)] = r_f$.

The controllability result of Proposition 13 is a consequence of the following weak observability result.

Proposition 14. *Let $T > 0$, $\psi_0 \in H^3_{(0)}(\Omega) \cap \mathcal{S}$ and $\psi_{ref}(t) := e^{i\Delta_D t}\psi_0$. We assume that **[H1]**, **[H2]** and **[H3]** hold. There exists $\mathcal{C}_1 > 0$ such that, for every $\phi_T \in H_0^1(\Omega, \mathbb{C})$ the solution of*

$$\begin{cases} \Big(i\partial_t + \Delta\Big)\phi(t,x) = 0\,, & (t,x) \in (0,T)\times\Omega\,, \\ \phi(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \\ \phi(T,x) = \phi_T\,, & x \in \Omega\,, \end{cases} \tag{27}$$

satisfies

$$\|\phi_T\|_{H_0^1(\Omega)} \leqslant \mathcal{C}_1 \left(\|2\Im(\partial_\nu \psi_{ref}\overline{\partial_\nu \phi}) + \mathcal{K}^*(i\phi)\|_{L^2((0,T)\times\Gamma_c)} + \|\phi_T\|_{H^{-1}(\Omega)}\right). \tag{28}$$

Proof of Proposition 13 thanks to Proposition 14: By [12, Theorems II.10 and II.19], the existence of a continuous right inverse to the continuous operator

$$\left| \begin{array}{llcl} F_K: & L^2((0,T)\times\Gamma_c,\mathbb{R}) & \to & \mathbb{P}_K[H^{-1}(\Omega)] \\ & G & \mapsto & \mathbb{P}_K[r(T)] \end{array} \right.$$

is equivalent to the existence of $\mathcal{C}_2>0$ such that

$$\|\phi_T\|_{H^1_0(\Omega)} \leqslant \mathcal{C}_2 \|F_K^*(\phi_T)\|_{L^2((0,T)\times\Gamma_c)}\,, \quad \forall \phi_T \in \mathbb{P}_K[H^1_0(\Omega)]\,, \tag{29}$$

where $F_K^*:\mathbb{P}_K[H^1_0(\Omega)]\to L^2((0,T)\times\Gamma_c,\mathbb{R})$ is the adjoint operator associated to F_K.

For $G\in L^2((0,T)\times\Gamma_c,\mathbb{R})$ and $\phi_T\in\mathbb{P}_K[H^1_0(\Omega)]$, we have (note that $\mathcal{K}$ is $\mathbb{R}$-linear)

$$\begin{array}{ll} & \Re\Big(\langle F_K(G),\overline{\phi_T}\rangle_{H^{-1},H^1_0}\Big) \\ = & \Re\Big(\langle r(T),\overline{\phi_T}\rangle_{H^{-1},H^1_0}\Big) \text{ because } \phi_T\in\mathbb{P}_K[H^1_0(\Omega)] \\ = & \Re\Big(-i\int_0^T\int_{\Gamma_c} 2G\partial_\nu\psi_{ref}\overline{\partial_\nu\phi}d\sigma(x)dt - i\int_0^T\int_\Omega\langle\mathcal{K}(G),\overline{\phi}\rangle_{H^{-1},H^1_0}dxdt\Big) \quad \text{by (25)} \\ = & \int_0^T\int_{\Gamma_c} G\Big(2\Im(\partial_\nu\psi_{ref}\overline{\partial_\nu\phi})+\mathcal{K}^*(i\phi)\Big)d\sigma(x)dt \quad \text{by (26)}\,. \end{array}$$

Thus F_K^* has the following explicit expression

$$\left| \begin{array}{llcl} F_K^*: & \mathbb{P}_K[H^1_0(\Omega)] & \to & L^2((0,T)\times\Gamma_c,\mathbb{R}) \\ & \phi_T & \mapsto & 2\Im(\partial_\nu\psi_{ref}\overline{\partial_\nu\phi})+\mathcal{K}^*(i\phi)\,. \end{array} \right.$$

Let $\mathcal{C}_1$ be as in Proposition 14. There exists $K\in\mathbb{N}^*$ such that

$$\mathcal{C}_1\|\phi_T\|_{H^{-1}(\Omega)} \leqslant \frac{1}{2}\|\phi_T\|_{H^1_0(\Omega)}\,, \quad \forall\phi_T\in\mathbb{P}_K[H^1_0(\Omega)]$$

and then (29) results from (28) with $\mathcal{C}_2:=2\mathcal{C}_1$. □

Proof of Proposition 14: Let $\widetilde{T}$ be as in **[H2]** and T',T'' be as in **[H3]**. Let $\epsilon:=(T''-T'-\widetilde{T})/2$ and $\rho\in C^\infty_c(\mathbb{R},\mathbb{R})$ be such that $\rho\equiv 1$ on $(T'+\epsilon,T'+\epsilon+\widetilde{T})$ and $\mathrm{Supp}(\rho)\subset(T',T'')$.

Step 1: We prove the existence of $C_1>0$ such that, for every $\phi_T\in H^1_0(\Omega)$, the solution of (27) satisfies

$$\|\phi_T\|_{H^1_0(\Omega)} \leqslant C_1\left(\|2\Im(\rho\partial_\nu\psi_{ref}\overline{\partial_\nu\phi})\|_{L^2(\mathbb{R}\times\Gamma_c)}+\|\phi_T\|_{H^{-1}(\Omega)}\right)\,. \tag{30}$$

Let $\phi_T\in H^1_0(\Omega)$. By **[H2]**, we have

$$\begin{array}{rcl} \|\phi_T\|^2_{H^1_0(\Omega)} & \leqslant & 2\mathcal{C}_0^2\left[\int\limits_{T'+\epsilon}^{T'+\epsilon+\widetilde{T}}\int\limits_{\Gamma_c}|\partial_\nu\phi(t,x)|^2d\sigma(x)dt+\|\phi_T\|^2_{H^{-1}}\right] \\ & \leqslant & 2\mathcal{C}_0^2\left[\int\limits_{\mathbb{R}}\int\limits_{\Gamma_c}\left|\rho(t)\overline{\partial_\nu\phi(t,x)}\right|^2d\sigma(x)dt+\|\phi_T\|^2_{H^{-1}}\right] \end{array}$$

because $\rho \equiv 1$ on $(T'+\epsilon, T'+\epsilon+\widetilde{T})$. Then, taking into account that $\mathrm{Supp}(\rho) \subset (T', T'')$ and $|\partial_\nu \psi_{ref}(t,x)| \geqslant m > 0$ for every $(t,x) \in (T',T'') \times \Gamma_c$, i.e **[H3]**, we obtain

$$\begin{array}{rcl}\|\phi_T\|^2_{H^1_0(\Omega)} & \leqslant & \frac{3\mathcal{C}_0^2}{m^2} \int_{\mathbb{R}} \int_{\Gamma_c} \left| \rho(t)\, \partial_\nu \psi_{ref}(t,x)\, \overline{\partial_\nu \phi(t,x)} \right|^2 d\sigma(x) dt \\ & & -\mathcal{C}_0^2 \int_{\mathbb{R}} \int_{\Gamma_c} |\rho(t)\, \partial_\nu \phi(t,x)|^2 d\sigma(x) dt + 2\mathcal{C}_0^2 \|\phi_T\|^2_{H^{-1}}\,.\end{array}$$

Moreover,

$$\begin{array}{rl} & 2 \int_{\mathbb{R}} \int_{\Gamma_c} \left|\rho\, \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi}\right|^2 d\sigma dt \\ = & \int_{\mathbb{R}} \int_{\Gamma_c} \left(\left|\rho\, \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi}\right|^2 + \left|\rho\, \partial_\nu \overline{\psi_{ref}}\, \partial_\nu \phi\right|^2 \right) d\sigma dt \\ = & \int_{\mathbb{R}} \int_{\Gamma_c} \left(\left|2\Im(\rho\, \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi})\right|^2 + 2\Re[(\rho \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi})^2] \right) d\sigma dt \end{array}$$

Note here that in the end of the proof, we will prove that the second term is compact. It will be crucial for that to notice that it is $\Re[(\rho \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi})^2]$ and not $\left|\Re[\rho \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi}]\right|^2$.

Thus

$$\begin{array}{rcl}\|\phi_T\|^2_{H^1_0(\Omega)} \leqslant & \frac{3\mathcal{C}_0^2}{2m^2} \int_{\mathbb{R}} \int_{\Gamma_c} \left(\left|2\Im(\rho\, \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi})\right|^2 + 2\Re[(\rho \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi})^2] \right) d\sigma dt \\ & -\mathcal{C}_0^2 \int_{\mathbb{R}} \int_{\Gamma_c} |\rho\, \partial_\nu \phi|^2 d\sigma dt + 2\mathcal{C}_0^2 \|\phi_T\|^2_{H^{-1}}\,.\end{array}$$

In order to get (30), it suffices to prove the existence of $C > 0$ such that

$$\int_{\mathbb{R}} \int_{\Gamma_c} \Re[(\rho \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi})^2] d\sigma dt \leqslant \frac{m^2}{3} \int_{\mathbb{R}} \int_{\Gamma_c} |\rho\, \partial_\nu \phi|^2 d\sigma dt + C \|\phi_T\|^2_{H^{-1}(\Omega)}\,. \tag{31}$$

Let $R \in \mathbb{N}^*$ that will be chosen later on,

$$\psi^H_{ref}(t) := \mathbb{P}_R[\psi_{ref}(t)] \quad \text{ and } \quad \psi^L_{ref} := \psi_{ref} - \psi^H_{ref}\,. \tag{32}$$

Then

$$\begin{array}{rl} & \int_{\mathbb{R}} \int_{\Gamma_c} \Re[(\rho\, \partial_\nu \psi_{ref}\, \overline{\partial_\nu \phi})^2] d\sigma dt \\ = & \int_{\mathbb{R}} \int_{\Gamma_c} \Re\Big((\rho\, \partial_\nu \psi^L_{ref}\, \overline{\partial_\nu \phi})^2 + (\rho\, \overline{\partial_\nu \phi})^2\, \partial_\nu \psi^H_{ref} (2\, \partial_\nu \psi^L_{ref} + \partial_\nu \psi^H_{ref}) \Big) d\sigma dt\,.\end{array}$$

In order to get (31), it is sufficient to prove that, for R large enough

$$\int_{\mathbb{R}} \int_{\Gamma_c} \Re[(\rho\, \partial_\nu \psi^L_{ref}\, \overline{\partial_\nu \phi})^2] d\sigma dt \leqslant C \|\phi_T\|_{H^{-1}}\,, \tag{33}$$

$$\int_{\mathbb{R}} \int_{\Gamma_c} \left| (\rho\, \overline{\partial_\nu \phi})^2\, \partial_\nu \psi^H_{ref} (2\, \partial_\nu \psi^L_{ref} + \partial_\nu \psi^H_{ref}) \right| d\sigma dt \leqslant \frac{m^2}{2} \int_{\mathbb{R}} \int_{\Gamma_c} |\rho\, \partial_\nu \phi|^2 d\sigma dt\,. \tag{34}$$

Note that $\psi^H_{ref} \to 0$ in $C^0([0,T], H^3_{(0)}(\Omega))$ when $R \to +\infty$ because $\psi_0 \in H^3_{(0)}(\Omega)$. Thus, $\partial_\nu \psi^H_{ref}\Big|_{\Gamma_c} \to 0$ in $L^\infty((0,T) \times \Gamma_c)$ when $R \to +\infty$, because of the Sobolev embeddings $\partial_\nu [H^3_{(0)}(\Omega)] \subset H^{3/2}(\Gamma_c) \subset L^\infty(\Gamma_c)$. As a consequence, there exists R such that

$$\|\partial_\nu \psi^H_{ref} (2 \partial_\nu \psi^L_{ref} + \partial_\nu \psi^H_{ref})\|_{L^\infty((0,T)\times\Gamma_c)} \leqslant \frac{m^2}{2}\,.$$

This implies (34). Until the end of Step 2, R is fixed. The function ρ is $C_c^\infty(\mathbb{R})$ thus, for every $N \in \mathbb{N}^*$, there exists $C_N > 0$ such that

$$\left| \int_{\mathbb{R}} \rho(t)^2 e^{i[\lambda_j + \lambda_J - \lambda_k - \lambda_K]t} dt \right| \leqslant \frac{C_N}{(\lambda_j + \lambda_J)^N}, \forall j, J \in \mathbb{N}^*, k, K \in \{1, ..., R-1\}.$$

We have

$$\begin{aligned}
& \left| \int_{\mathbb{R}} \int_{\Gamma_c} \Re[(\rho \partial_\nu \psi_{ref}^L \overline{\partial_\nu \phi})^2] d\sigma dt \right| \\
= & \left| \sum_{k,K=1}^{R-1} \sum_{j,J=1}^{\infty} \psi_k^0 \psi_K^0 \overline{\phi_j^0 \phi_J^0} \left(\int_{\mathbb{R}} \rho(t)^2 e^{i[\lambda_j + \lambda_J - \lambda_k - \lambda_K]t} dt \right) \left(\int_{\Gamma_c} \partial_\nu \varphi_k \, \partial_\nu \varphi_K \, \partial_\nu \varphi_j \, \partial_\nu \varphi_J d\sigma \right) \right| \\
\leqslant & \sum_{k,K=1}^{R-1} \sum_{j,J=1}^{\infty} |\psi_k^0 \psi_K^0 \phi_j^0 \phi_J^0| \tfrac{C_N}{(\lambda_j + \lambda_J)^N} \lambda_k^\alpha \lambda_K^\alpha \lambda_j^\alpha \lambda_J^\alpha \quad \text{ for some } \alpha > 0 \\
\leqslant & \; C(\psi_0, R, N) \left(\sum_j^\infty \lambda_j^{\alpha - N} |\phi_j^0| \right)^2 \\
\leqslant & \; C'(\psi_0, R, N) \|\phi_T\|_{H^{-1}(\Omega)}^2 \text{ for } N \text{ large enough}.
\end{aligned}$$

Here, we have used Weyl law so get that for another N large enough, $\sum_j^\infty \lambda_j^{-N} < +\infty$. Inequality (33) is proved, which ends Step 1.

Step 2: We prove the existence of $C_2 > 0$ such that, for every $\phi_T \in H_0^1(\Omega)$, the solution of (27) satisfies

$$\|\phi_T\|_{H_0^1(\Omega)} \leqslant C_2 \left(\|2\Im(\partial_\nu \psi_{ref} \overline{\partial_\nu \phi}) + \mathcal{K}^*(i\phi)\|_{L^2((0,T)\times\Gamma_c)} + \|\phi_T\|_{H^{-1}(\Omega)} \right). \tag{35}$$

>From (30), we deduce that

$$\|\phi_T\|_{H_0^1(\Omega)} \leqslant C_1 \left(\|2\Im(\rho \partial_\nu \psi_{ref} \overline{\partial_\nu \phi}) + \mathcal{K}^*[i\phi]\|_{L^2(\mathbb{R}\times\Gamma_c)} + \|\mathcal{K}^*[i\phi]\|_{L^2(\mathbb{R}\times\Gamma_c)} + \|\phi_T\|_{H^{-1}(\Omega)} \right).$$

In order to prove (35), it suffices to prove the existence of $C > 0$ such that,

$$\|\mathcal{K}^*[i\phi]\|_{L^2((0,T)\times\Gamma_c)} \leqslant \frac{1}{2C_1} \|\phi_T\|_{H_0^1(\Omega)} + C\|\phi_T\|_{H^{-1}}, \quad \forall \phi_T \in H_0^1(\Omega). \tag{36}$$

Let $R \in \mathbb{N}^*$ that will be chosen later on and ψ_{ref}^H be defined by (32). >From (24) and expansion (20), we see that $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$ where

$$\left| \begin{array}{rccc} \mathcal{K}_1 : & L^2((0,T) \times \Gamma_c, \mathbb{R}) & \to & L^2((0,T), H^{-1}(\Omega)) \\ & G & \mapsto & \Delta^2[V\psi_{ref}^H] \end{array} \right.$$

and

$$\left| \begin{array}{rccc} \mathcal{K}_2 : & L^2((0,T) \times \Gamma_c, \mathbb{R}) & \to & L^2((0,T), H^{-1/2-\varepsilon}(\Omega)) \\ & G & \mapsto & \Delta^2[V\psi_{ref}^L] \end{array} \right.$$

is continuous. We denote by $\|\mathcal{K}_1\|$ and $\|\mathcal{K}_2\|$ their operator norm in this functional frame.

Note that $\psi_{ref}^H \to 0$ in $C^0([0,T], H^3_{(0)}(\Omega))$ when $R \to +\infty$ because $\psi_0 \in H^3_{(0)}(\Omega)$. Thus, there exists $R \in \mathbb{N}^*$ such that

$$\|\mathcal{K}_1\| \leqslant \frac{1}{2C_1\sqrt{T}}.$$

>From now on R is fixed. Then, for every $\phi_T \in H^1_0(\Omega)$,

$$\begin{aligned}\|\mathcal{K}^*[i\phi]\|_{L^2((0,T)\times\Gamma_c)} &\leqslant \|\mathcal{K}_1^*[i\phi]\|_{L^2((0,T)\times\Gamma_c)} + \|\mathcal{K}_2^*[i\phi]\|_{L^2((0,T)\times\Gamma_c)} \\ &\leqslant \frac{1}{2C_1\sqrt{T}}\|\phi\|_{L^2((0,T),H^1_0(\Omega))} + \|\mathcal{K}_2\|\|\phi\|_{L^2((0,T),H^{1/2+\varepsilon}(\Omega))} \\ &\leqslant \frac{1}{2C_1}\|\phi_T\|_{H^1_0(\Omega)} + \sqrt{T}\|\mathcal{K}_2\|\|\phi_T\|_{H^{1/2+\varepsilon}(\Omega)}\,.\end{aligned}$$

This implies (36) after interpolation, which ends Step 2. □

4.2 Control of high frequencies for the nonlinear system

The goal of this section is the proof of Theorem 1. Thus, in the whole section, $T > 0$, $\psi_0 \in H^3_{(0)}(\Omega) \cap \mathcal{S}$, $\psi_{ref}(t) := e^{i\Delta_D t}\psi_0$ are fixed and **[H1]**, **[H2]**, **[H3]** are assumed to hold. We consider the end-point map

$$\left|\begin{array}{rcl} \widetilde{\Theta}_K : \quad L^2((0,T)\times\Gamma_c,\mathbb{R}) & \to & \mathbb{P}_K[H^3_{(0)}(\Omega)] \cap \mathcal{S} \\ g & \mapsto & \mathbb{P}_K[\psi(T)] \end{array}\right.$$

where ψ solves (1). By Proposition 10, $\widetilde{\Theta}_K$ is of class C^1 and

$$\left|\begin{array}{rcl} d\widetilde{\Theta}_K(0) : \quad L^2((0,T)\times\Gamma_c,\mathbb{R}) & \to & \mathbb{P}_K[H^3_{(0)}(\Omega)] \cap T_{\mathcal{S}}\psi_{ref}(T) \\ G & \mapsto & \mathbb{P}_K[\Psi(T)] \end{array}\right.$$

where Ψ solves (22) with $\Psi_0 = 0$. By Proposition 11, there exists $K \in \mathbb{N}^*$ such that $d\widetilde{\Theta}_K(0)$ has a continuous right inverse. By the inverse mapping theorem, $\widetilde{\Theta}_K$ is a local C^1-diffeomorphism on a neighborhood of 0. The first statement of Theorem 1 holds with $\Upsilon := \widetilde{\Theta}_K^{-1}$ which is locally well defined.

There exists $K' \geqslant K$ such that

$$\|\psi_{ref}(T) - \mathbb{P}_{K'}[\psi_{ref}(T)]\|_{H^3_{(0)}} < \delta\,.$$

Then, $\psi_f := (\mathbb{P}_K - \mathbb{P}_{K'})[\psi_{ref}(T)]]$ belongs to $\mathcal{V}$ thus $g := \Upsilon(\psi_f)$ is well defined in $L^2((0,T)\times\Gamma_c,\mathbb{R})$ and the associated solution of (1) satisfies $\mathbb{P}_K[\psi(T)] = \psi_f$ thus $\mathbb{P}_{K'}[\psi(T)] = 0$. As a consequence $\psi(T)$ is a finite sum of eigenfunctions of $(-\Delta_D)$ so it is a smooth function. □

5 Local exact control around eigenfunctions

The goal of this section is to prove Theorem 2, by following essentially the same strategy as in the previous section. In Section 5.1, we prove the observability of the adjoint of the linearized system. In Section 5.2, we prove Theorem 2.

5.1 Observability results

The goal of this section is to prove the following observability inequality.

Proposition 15. *Let $T > 0$, $R \in \mathbb{N}^*$. We assume that **[H1]**, **[H2]**, **[H3']** and **[H4]** hold. There exists $\mathcal{C}_2 > 0$ such that, for every $\phi_T \in H^1_0(\Omega) \cap T_{\mathcal{S}}\varphi_R$ the solution of*

$$\begin{cases} \Big(i\partial_t + \Delta + \lambda_R\Big)\phi(t,x) = 0\,, & (t,x) \in (0,T)\times\Omega\,, \\ \phi(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \\ \phi(T,x) = \phi_T\,, & x \in \Omega\,, \end{cases} \tag{37}$$

satisfies

$$\|\phi_T\|_{H^1_0(\Omega)} \leqslant \mathcal{C}_2 \|2\Im(\partial_\nu \varphi_R \overline{\partial_\nu \phi}) + \widetilde{\mathcal{K}}^*(i\phi)\|_{L^2((0,T)\times\Gamma_c)}\,, \tag{38}$$

where

$$\left| \begin{array}{llll} \widetilde{\mathcal{K}}: & L^2(\Gamma_c,\mathbb{R}) & \to & L^2(\Omega,\mathbb{C}) \\ & G & \mapsto & \Delta^2[V\varphi_R] \quad \textit{where} \left\{ \begin{array}{l} (-\Delta+1)V = 0\,,\ \textit{in}\ \Omega\,, \\ \partial_\nu V = G 1_{\Gamma_c}\ \textit{on}\ \partial\Omega\,. \end{array} \right. \end{array} \right. \tag{39}$$

The proof of Proposition 15 relies on 2 key ingredients: the weak observability of the Schrödinger equation **[H2]** and a unique continuation result given by the following statement.

Proposition 16. *Let $T>0$, $R\in\mathbb{N}^*$. We assume that **[H1]**, **[H2]**, **[H3']** and **[H4]** hold. Then for every non zero $\phi_T \in H^1_0(\Omega)\cap T_{\mathcal{S}}\varphi_R$,*

$$2\Im(\partial_\nu \varphi_R \overline{\partial_\nu \phi}) + \widetilde{\mathcal{K}}^*(i\phi) \neq 0 \quad \textit{on}\ (0,T)\times\Gamma_c\,,$$

where $\phi(t) := e^{i\Delta_D(t-T)}\phi_T$.

Proof of Proposition 16: Let $T' \in (\widetilde{T}, T)$ where $\widetilde{T}$ is as in **[H2]**. By Proposition 14 (with T replaced by T') and the change of phase $\phi(t,x) \leftarrow \phi(t,x)e^{i\lambda_R t}$, we know that there exists $\mathcal{C}_1>0$ such that, for every $\phi_T \in H^1_0(\Omega)$, the solution of (37) satisfies

$$\|\phi_T\|_{H^1_0(\Omega)} \leqslant \mathcal{C}_1\Big(\|2\Im(\partial_\nu \varphi_R \overline{\partial_\nu \phi}) + \widetilde{\mathcal{K}}^*(i\phi)\|_{L^2((0,T')\times\Gamma_c)} + \|\phi_T\|_{H^{-1}(\Omega)}\Big)\,. \tag{40}$$

We introduce

$$N_T := \Big\{ \phi_0 \in H^1_0(\Omega)\cap T_{\mathcal{S}}\varphi_R; 2\Im(\partial_\nu \varphi_R \overline{\partial_\nu \phi}) + \widetilde{\mathcal{K}}^*(i\phi) = 0 \text{ in } L^2((0,T)\times\Gamma_c)\Big\}$$

where $\phi(t) := e^{i(\Delta_D+\lambda_R)t}\phi_0 := S(t)\phi_0$. N_T is a $\mathbb{R}$-vector subspace of $H^1_0(\Omega)$. We want to prove that $N_T=\{0\}$.

Step 1: We prove that N_T contains eigenfunctions of $-\Delta_D$. Let $\phi_0 \in N_T$. Since $\phi_0 \in H^1_0(\Omega)$, then $\phi_\epsilon := \frac{S(\epsilon)\phi_0 - \phi_0}{\epsilon}$ is bounded in $H^{-1}(\Omega)$ uniformly when $\epsilon \to 0$. Moreover, for $\epsilon < T-T'$, ϕ_ϵ belongs to $N_{T'}$. Applying (40) to ϕ_ϵ, with $\epsilon < T-T'$, we get $\|\phi_\epsilon\|_{H^1_0(\Omega)} \leqslant \mathcal{C}_1 \|\phi_\epsilon\|_{H^{-1}}$. Thus ϕ_ϵ is bounded in $H^1_0(\Omega)$ when $\epsilon\to 0$. Therefore $\phi_0 \in H^3_{(0)}(\Omega)$. Indeed, by Fatou lemma,

$$\begin{array}{ll} \|\phi_0\|^2_{H^3_{(0)}} & = \sum_{k=1}^\infty \Big|\lambda_k^{3/2}\langle \phi_0,\varphi_k\rangle\Big|^2 \\ & = \sum_{k=1}^\infty \liminf_{\epsilon\to 0} \Big|\sqrt{\lambda_k}\frac{e^{i\lambda_k\epsilon}-1}{\epsilon}\langle \phi_0,\varphi_k\rangle\Big|^2 \\ & \leqslant \liminf_{\epsilon\to 0}\sum_{k=1}^\infty \Big|\sqrt{\lambda_k}\frac{e^{i\lambda_k\epsilon}-1}{\epsilon}\langle \phi_0,\varphi_k\rangle\Big|^2 \\ & \leqslant \liminf_{\epsilon\to 0} \|\phi_\epsilon\|_{H^1_0} < \infty\,. \end{array}$$

For $T''<T$, the map

$$\left| \begin{array}{llll} \mathcal{J}_{T''}: & H^1_0(\Omega) & \to & L^2((0,T'')\times\Gamma_c) \\ & \phi_0 & \mapsto & 2\Im[\partial_\nu\varphi_R\overline{\partial_\nu\phi}] + \mathcal{K}^*(i\phi) \end{array} \right.$$

is continuous (see Propositions 8 and 9, which implies the continuity of $\mathcal{K} : L^2(\Gamma_c, \mathbb{R}) \to L^2(\Omega)$ and thus of $\mathcal{K}^* : L^2(\Omega) \mapsto L^2(\Gamma_c, \mathbb{R})$). Moreover, $\mathcal{J}_{T''}(\phi_\epsilon) = 0$ for every $\epsilon < T - T''$, and $\phi_\epsilon \to i(\Delta_D + \lambda_R)\phi_0$ in $H_0^1(\Omega)$ thus $\mathcal{J}_{T''}[i(\Delta_D + \lambda_R)\phi_0] = 0$. This holds for every $T'' \in (0, T)$, thus $i(\Delta_D + \lambda_R)\phi_0 \in N_T$.

We have proved that N_T is stable by $i(\Delta_D + \lambda_R)$ and only contains smooth functions in $D[(-\Delta_D)^s]$, $\forall s \in \mathbb{N}^*$. Applying again estimate (40) to $i(\Delta_D + \lambda_R)\phi_0$, we get

$$\|(\Delta_D + \lambda_R)\phi_0\|_{H_0^1(\Omega)} \leqslant \mathcal{C}_1 \|(\Delta_D + \lambda_R)\phi_0\|_{H^{-1}(\Omega)} .$$

This shows that the unit ball of N_T is compact (for the $H_0^1(\Omega)$-topology), thus N_T has finite dimension (Rellich theorem).

Then the finite dimensional space N_T is stable by the real symmetric operator $-(\Delta_D + \lambda_R)^2 = [i(\Delta_D + \lambda_R)]^2$, thus $-(\Delta_D + \lambda_R)^2$ has an eigenfunction in N_T. This candidate is also an eigenfunction of $(-\Delta_D)$.

Step 2: We prove that

$$\widetilde{\mathcal{K}}^*(z\varphi_k) = \Re(z)\Big(\lambda_k^2 w_k - 2\,\partial_\nu\varphi_R\,\partial_\nu\varphi_k\Big)_{|\Gamma_c}, \quad \forall z \in \mathbb{C}, k \in \mathbb{N}^* \tag{41}$$

where w_k is defined in **[H4]**. Let $z \in \mathbb{C}$ and $k \in \mathbb{N}^*$. For every $G \in L^2(\Gamma_c, \mathbb{R})$, we have

$$\int_{\Gamma_c} G\widetilde{\mathcal{K}}^*(z\varphi_k)d\sigma(x) = \Re\left(\int_\Omega \widetilde{\mathcal{K}}(G)\overline{z\varphi_k}dx\right) = \Re(z)\int_\Omega \widetilde{\mathcal{K}}(G)\varphi_k dx$$

thus $\widetilde{\mathcal{K}}^*(z\varphi_k) = \Re(z)\widetilde{\mathcal{K}}^*(\varphi_k)$.

If $G \in C_c^\infty((0,T) \times \Gamma_c, \mathbb{R})$, then the potential V defined in (39) belongs to $C^\infty((0,T) \times \Omega, \mathbb{R})$ thus the following integrations by part are legitimate

$$\begin{aligned} & \textstyle\int_\Omega \widetilde{\mathcal{K}}(G)\varphi_k dx = \int_\Omega \Delta^2[V\varphi_R]\varphi_k dx \text{ with } V \text{ as in (39)} \\ = \; & \textstyle\int_{\partial\Omega}\Big(\partial_\nu[\Delta(V\varphi_R)]\varphi_k - \Delta(V\varphi_R)\,\partial_\nu\varphi_k\Big)d\sigma(x) + \int_\Omega \Delta[V\varphi_R]\Delta\varphi_k dx \\ = \; & \textstyle -2\int_{\Gamma_c} G\partial_\nu\varphi_R\,\partial_\nu\varphi_k\,d\sigma(x) - \lambda_k\int_\Omega \Delta[V\varphi_R]\varphi_k dx\,, \end{aligned}$$

$$\begin{aligned} \textstyle\int_\Omega \Delta[V\varphi_R]\varphi_k dx & \textstyle = -\lambda_k\int_\Omega V\varphi_R\varphi_k dx \\ & \textstyle = -\lambda_k\int_\Omega V(-\Delta+1)w_k dx \\ & \textstyle = -\lambda_k\int_{\Gamma_c} Gw_k d\sigma(x)\,. \end{aligned}$$

This proves that

$$\int_\Omega \widetilde{\mathcal{K}}(G)\varphi_k dx = \int_{\Gamma_c} G\left(\lambda_k^2 w_k - 2\partial_\nu\varphi_R\,\partial_\nu\varphi_k\right)d\sigma$$

for every $G \in C_c^\infty((0,T) \times \Gamma_c, \mathbb{R})$. The same equality holds for every $G \in L^2((0,T) \times \Gamma_c, \mathbb{R})$ by density. Therefore, $\widetilde{\mathcal{K}}^*(\varphi_k) = \lambda_k^2 w_k - 2\partial_\nu\varphi_R\,\partial_\nu\varphi_k$.

Step 3: We prove that $N_T = \{0\}$. Working by contradiction, we assume that $N_T \neq \{0\}$. By Step 1, there exists $z \in \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}^*$ such that $z\varphi_k \in N_T$. Moreover, by Step 2,

$$2\Im\left(\partial_\nu\varphi_R\overline{\partial_\nu[z\varphi_k e^{i(\lambda_R-\lambda_k)t}]}\right) + \widetilde{\mathcal{K}}^*(iz\varphi_k e^{i(\lambda_R-\lambda_k)t}) = \Re[ize^{i(\lambda_R-\lambda_k)t}]\lambda_k^2 w_k\,.$$

Thus $\Re[ize^{i(\lambda_R-\lambda_k)t}]w_k$ vanishes on $(0,T)\times\Gamma_c$. By assumption **[H4]**, we deduce that

$$\Re[ize^{i(\lambda_R-\lambda_k)t}]\equiv 0 \quad \text{on } (0,T)\,. \tag{42}$$

First case: $k=R$. Then $z\in i\mathbb{R}$ because $z\varphi_k\in T_{\mathcal{S}}\varphi_R$. >From (42), we deduce that $0=\Re(iz)=iz$, which is impossible because $z\neq 0$.
Second case: $k\neq R$. Then $\lambda_k\neq\lambda_R$ because λ_R is a simple eigenvalue (see **[H4]**). >From (42), we deduce that $z=0$, which is a contradiction. □

Remark 2. *Note that assumption **[H4]** is necessary for the controllability of the linearized system 22 (with $\Psi_0=0$) around $(\psi_{ref}(t)=\varphi_R e^{-i\lambda_R t}, g_{ref}\equiv 0)$. Indeed, for any eigenfunction Φ of $(-\Delta_D)$ associated with the eigenvalue λ_Φ, we have*

$$\begin{aligned}\langle\Psi(T),\Phi\rangle &= \left\langle i\int_0^T e^{-i\Delta_D(T-t)}\Big(V(t)\psi_{ref}(t)\Big)dt,\Phi\right\rangle\\ &= ie^{-i\lambda_\Phi T}\int_0^T e^{-i(\lambda_R-\lambda_\Phi)t}\langle V(t),\varphi_R\Phi\rangle dt\\ &= ie^{-i\lambda_\Phi T}\int_0^T e^{-i(\lambda_R-\lambda_\Phi)t}\langle G(t),w\rangle_{L^2(\Gamma_c)}dt\end{aligned}$$

*where w is as in **[H4]** (apply the Green formula, as in the proof of Proposition 2). In particular, if **[H4]** is not fulfilled, we are in one of the following situation*

- *if λ_R is a multiple eigenvalue, then, for any eigenfunction Φ associated to the eigenvalue $\lambda_\Phi=\lambda_R$ and linearly independent of φ_R, one cannot control the complex component $i\langle\Psi(T),\Phi\rangle e^{i\lambda_\Phi T}$ because it only takes real values*
$$i\langle\Psi(T),\Phi\rangle e^{i\lambda_\Phi T}=\int_0^T\langle G(t),w\rangle dt\in\mathbb{R}\,,$$
- *if there exists an eigenfunction Φ for wich $w\equiv 0$ on Γ_c, then, one cannot control the component $\langle\Psi(T),\Phi\rangle$ because it vanishes*
$$\langle\Psi(T),\Phi\rangle = ie^{-i\lambda_\Phi T}\int_0^T e^{-i(\lambda_R-\lambda_\Phi)t}\langle G(t),w\rangle_{L^2(\Gamma_c)}dt=0\,.$$

Proof of Proposition 15: Working by contradiction, we assume that, for every $n\in\mathbb{N}^*$, there exists $\phi_T^n\in H_0^1(\Omega)$ such that

$$1=\|\phi_T^n\|_{H_0^1(\Omega)}>n\|2\Im(\partial_\nu\varphi_R\overline{\partial_\nu\phi^n})+\widetilde{\mathcal{K}}^*(i\phi^n)\|_{L^2((0,T)\times\Gamma_c)}\,. \tag{43}$$

Up to a subsequence, one may assume that $\phi_T^n\rightharpoonup\phi_T^\infty$ weakly in $H_0^1(\Omega)$. The strong continuity of the operator

$$\left|\begin{array}{ccc} H_0^1(\Omega) & \longrightarrow & L^2((0,T)\times\Gamma_c)\\ \phi_T & \longmapsto & 2\Im(\partial_\nu\varphi_R\overline{\partial_\nu\phi})+\widetilde{\mathcal{K}}^*(i\phi)\end{array}\right.$$

implies its continuity for the weak topology [12, Theorem III.9]. Thus

$$2\Im(\partial_\nu\varphi_R\overline{\partial_\nu\phi^n})+\widetilde{\mathcal{K}}^*(i\phi^n)\underset{n\to\infty}{\rightharpoonup}2\Im(\partial_\nu\varphi_R\overline{\partial_\nu\phi^\infty})+\widetilde{\mathcal{K}}^*(i\phi^\infty)\text{ in }L^2((0,T)\times\Gamma_c)\,.$$

>From (43), we deduce that

$$2\Im(\partial_\nu\varphi_R\overline{\partial_\nu\phi^\infty})+\widetilde{\mathcal{K}}^*(i\phi^\infty)=0\text{ on }(0,T)\times\Gamma_c\,.$$

By Proposition 16, $\phi_T^\infty = 0$. Thus, up to a subsequence, $\|\phi_T^n\|_{H^{-1}(\Omega)} \to 0$ when $n \to \infty$. We deduce from (40) that

$$1 = \|\phi_T^n\|_{H_0^1(\Omega)} \leqslant \mathcal{C}_1\Big(\frac{1}{n} + \|\phi_T^n\|_{H^{-1}(\Omega)}\Big) \underset{n\to\infty}{\to} 0\,,$$

which is a contradiction. □

5.2 Proof of Theorem 2

Let $T > 0$, $R \in \mathbb{N}^*$ and $\psi_{ref}(t) := \varphi_R(x)e^{-i\lambda_R t}$. We assume that **[H1]**, **[H2]**, **[H3']** and **[H4]** hold. By Proposition 10, the end point map

$$\left|\begin{array}{lccclcl} \widetilde{\Theta}: & [H^3_{(0)}(\Omega)\cap\mathcal{S}] & \times & L^2((0,T)\times\Gamma_c,\mathbb{R}) & \to & [H^3_{(0)}(\Omega)\cap\mathcal{S}]^2 \\ & (\psi_0 & , & g) & \mapsto & (\psi_0, \psi(T)e^{i\lambda_R T}) \end{array}\right.$$

is of class C^1 and

$$\left|\begin{array}{lccclcl} d\widetilde{\Theta}(\varphi_R,0): & [H^3_{(0)}(\Omega)\cap T_{\mathcal{S}}\varphi_R] & \times & L^2((0,T)\times\Gamma_c,\mathbb{R}) & \to & [H^3_{(0)}(\Omega)\cap T_{\mathcal{S}}\varphi_R]^2 \\ & (\Psi_0 & , & G) & \mapsto & (\Psi_0, \Psi(T)) \end{array}\right.$$

where

$$\begin{cases} \Big(i\partial_t + \Delta + \lambda_R\Big)\Psi(t,x) = V(t,x)\varphi_R\,, & (t,x)\in(0,T)\times\Omega\,, \\ \Psi(t,x) = 0\,, & (t,x)\in(0,T)\times\partial\Omega\,, \\ \Psi(0,x) = \Psi_0(x)\,, & x\in\Omega\,, \\ (-\Delta+1)V(t,x) = 0\,, & (t,x)\in(0,T)\times\Omega\,, \\ \partial_\nu V(t,x) = G(t,x)1_{\Gamma_c}(x)\,, & (t,x)\in(0,T)\times\partial\Omega\,. \end{cases} \tag{44}$$

The same arguments as in the previous section prove that $d\widetilde{\Theta}(\varphi_R, 0)$ has a continuous right inverse thanks to Proposition 15. The inverse mapping theorem, proves that $\widetilde{\Theta}$ is a local C^1-diffeomorphism. □

5.3 Some comments about control only depending on time

In this subsection, we comment on the information given by our previous study for a control that would be of the form

$$\begin{cases} \Big(i\partial_t + \Delta\Big)\psi(t,x) = v(t,x)\psi(t,x)\,, & (t,x)\in(0,T)\times\Omega\,, \\ \psi(t,x) = 0\,, & (t,x)\in(0,T)\times\partial\Omega\,, \\ \psi(0,x) = \psi_0(x) & x\in\Omega\,, \\ (-\Delta+1)v(t,x) = 0\,, & (t,x)\in(0,T)\times\Omega\,, \\ \partial_\nu v(t,x) = g(t)\mu(x)1_{\Gamma_c}(x) & (t,x)\in(0,T)\times\partial\Omega\,, \end{cases} \tag{45}$$

where the control $g \in L^2(0,T)$ only depends on time and $\mu \in L^\infty(\partial\Omega)$ is fixed.

This is very close to the configuration studied in several papers described in the introduction (except that we impose that the potential $v = g(t)V_\mu$ is harmonic).

If we followed the framework described in the previous section, we would obtain that the observability estimate necessary to obtain the controllability of the linearized system, would be of the form

$$\|\phi_T\|^2_{H^1_0(\Omega)} \leqslant \mathcal{C}_2 \int_0^T \left| \int_{\partial\Omega} 2\Im(\mu(x)\partial_\nu \varphi_R \overline{\partial_\nu \phi}) + \mathcal{K}^*(i\phi) d\sigma(x) \right|^2 dt\,, \tag{46}$$

instead of the classical observability estimate (38), where $\mathcal{K}$ would be another (but similar) compact operator.

Then, it becomes quite clear that (46) is really not likely to be true in general. Indeed, to contradict (46), it suffices to find a sequence of solutions with H^1_0 norm equals to 1 but whose Neumann trace on the boundary weakly converges to zero. This can be easily done for instance

- on the square $(0,\pi)^2$: with the normalized initial data $\phi_n(x,y) = c_n \sin(nx)\sin(ny)$
- on the disk: with some initial data $\phi_n(r,\theta) = e^{in\theta} g_n(r)$ where g_n are appropriate normalized Bessel functions.

Also, this analysis can also help to identify some class of initial data that could be observable. It is necessary that there is no oscillation of the trace on the boundary. This could be true for instance

- on square: with some initial data that would depend only on x or y.
- on the disk: with some radial initial data.

Also, from a microlocal point of view, we can see that (46) can only be true for sequences of solutions whose trace do not weakly converge to zero. That means some data whose Neumann trace on the boundary is concentrated in the frequency $\xi' = 0$. Yet, the theorems of propagation of microlocal defect measure (described for example for the wave equation in [14]) could suggest that it implies that the data are concentrated close to some rays that intersect the boundary orthogonally.

6 Schrödinger equation with real-valued boundary controls

The goal of this section is to prove Theorem 3. The arguments have already been developped in the Step 1 of the proof of Proposition 14 but here, we give a simpler proof in this simpler case.

Proposition 17. *Let Ω be an open subet of $\mathbb{R}^2$, Γ be an open subset of $\partial\Omega$ and $0 < \widetilde{T} < T < \infty$. If the Schrödinger equation on Ω is weakly observable on $(0,\widetilde{T}) \times \Gamma$ then, there exists $\mathcal{C}_1 > 0$ such that*

$$\|\phi_0\|_{H^1_0(\Omega)} \leqslant \mathcal{C}_1 \Big(\|\Im(\partial_\nu \phi)\|_{L^2((0,T)\times\Gamma)} + \|\phi_0\|_{H^{-1}(\Omega)} \Big)\,, \quad \forall \phi_0 \in H^1_0(\Omega,\mathbb{C}) \tag{47}$$

where $\phi(t) := e^{it\Delta_D}\phi_0$.

Proof. Let $T', T'' \in (0,T)$ and $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R}^+)$ be such that $T''-T' > \widetilde{T}$, $\rho \equiv 1$ on (T', T'') and $\mathrm{Supp}(\rho) \subset (0,T)$. The assumption and conservation of the norm give

$$\|\phi_0\|^2_{H^1_0(\Omega)} \leqslant \mathcal{C}_0\Big(\|\rho\partial_\nu \phi\|^2_{L^2(\mathbb{R}_t\times\Gamma)} + \|\phi_0\|^2_{H^{-1}(\Omega)}\Big)\,, \quad \forall \phi_0 \in H^1_0(\Omega,\mathbb{C}) \tag{48}$$

Since $|\Im(\partial_\nu\phi)|^2 = \frac{|\partial_\nu\phi|^2}{2} + \frac{\Re\left[(\partial_\nu\phi)^2\right]}{2}$, it is enough to prove

$$\left| \int_{\mathbb{R}_t}\int_\Gamma \rho^2(\partial_\nu\phi)^2 \right| \leq C\|\phi_0\|^2_{H^{-1}(\Omega)}.$$

This inequality is obviously false if $(\partial_\nu\phi)^2$ is replaced by $|\partial_\nu\phi|^2$. It is actually true because $(\partial_\nu\phi)^2$ is the product of two terms that oscillate in the same direction, which is false for $|\partial_\nu\phi|^2$.

More precisely, write $\phi_0 = \sum_k \varphi_k\phi_{0,k}$ where $(\varphi_k)_{k\in\mathbb{N}^*}$ is an orthonormal basis of eigenfunctions for Δ_D with eigenvalues $-\lambda_k$. We have $\phi = \sum_k e^{-i\lambda_k}\varphi_k\phi_{0,k}$

$$\int_{\mathbb{R}_t}\int_\Gamma \rho^2(\partial_\nu\phi)^2 = \sum_{k,l}\phi_{0,l}\phi_{0,k}\left(\int_{\mathbb{R}_t} e^{-i(\lambda_k+\lambda_l)}\rho^2\right)\left(\int_\Gamma(\partial_\nu\varphi_k)(\partial_\nu\varphi_l)\right)$$

The boundary term can be bounded for instance by trace estimates (actually, finner estimates would give a better exponent $\lambda_k^{1/2}$)

$$\begin{aligned}\textstyle\int_\Gamma |(\partial_\nu\varphi_k)(\partial_\nu\varphi_l)| \quad &\leq \|\partial_\nu\varphi_k\|_{L^2(\Gamma)}\|\partial_\nu\varphi_l\|_{L^2(\Gamma)} \leq C\,\|\varphi_k\|_{H^2(\Omega)}\,\|\varphi_l\|_{H^2(\Omega)}\\ &\leq C\lambda_k\lambda_l \leq C(\lambda_k+\lambda_l)^2.\end{aligned}$$

The function ρ is $C_c^\infty(\mathbb{R})$ thus, for every $N\in\mathbb{N}^*$, there exists $C_N>0$ such that

$$\left|\int_{\mathbb{R}}\rho(t)^2 e^{-i[\lambda_k+\lambda_l]t}dt\right| \leqslant \frac{C_N}{(\lambda_k+\lambda_l)^N} \leq \frac{\widetilde{C_N}}{(\lambda_k\lambda_l)^{N/2}}\,, \forall k,l\in\mathbb{N}^*$$

So, choosing N large enough (and replacing $N/2-2$ by N), we obtain the bound

$$\left|\int_{\mathbb{R}_t}\int_\Gamma\rho^2(\partial_\nu\phi)^2\right| \leq \sum_{k,l}|\phi_{0,l}\phi_{0,k}|\frac{C_N}{(\lambda_k\lambda_l)^N} \leq C_N\left(\sum_k\frac{|\phi_{0,k}|}{\lambda_k^N}\right)^2 \leq C\|\phi_0\|^2_{H^{-1}(\Omega)}$$

when N is chosen large enough so that $\lambda_k^{1/2-N}$ is summable, which is always possible (for instance using Weyl law). □

The following result can be proved thanks to the arguments developped in the proof of Proposition 16 (in a simpler way, because $\mathcal{K}^*$ does not appear anymore), together with Proposition 17 and the time-oscillation of $\partial_\nu\phi$.

Proposition 18. *Let Ω be an open subet of $\mathbb{R}^2$, Γ be an open subset of $\partial\Omega$ and $0<\widetilde{T}<T<\infty$. If the Schrödinger equation on Ω is weakly observable on $(0,\widetilde{T})\times\Gamma$ then, for every non zero $\phi_0\in H^1_0(\Omega)\cap T_{\mathcal{S}}\varphi_R$,*

$$\Im(\partial_\nu\phi)\neq 0 \quad \text{on } (0,T)\times\Gamma_c\,,$$

where $\phi(t) := e^{it\Delta_D}\phi_0$.

Proof. Since the method is classical and was already performed in a more complicated case, we only detail the difference with Proposition 16. We define similarly

$$N_T := \{\phi_0 \in H^1_0(\Omega); \Im(\partial_\nu \phi) = 0 \text{ in } L^2((0,T)\times\Gamma_c)\}$$

where $\phi(t) := e^{i\Delta_D t}\phi_0$.

Following Step 1, we obtain that if $N_T \neq \{0\}$, N_T only contains one eigenfunction. Assume $\varphi \in N_T \setminus \{0\}$ is an eigenfunction of Δ_D. Then, $e^{it\Delta_D}\varphi = e^{-it\lambda}\varphi$. Note that the Dirichlet boundary condition implies $\lambda \neq 0$ and therefore $\varphi \in N_T$ implies $\partial_\nu\varphi = 0$ on Γ_c. By unique continuation for eigenfunctions, we get $\varphi = 0$, a contradiction. □

Remark 3. *This result would be false if we considered Neumann boundary conditions. The real constants solutions $\Phi(t) = c$ with $c \in \mathbb{R}$ create a zero observation. The system is therefore not controllable with a real control. Yet, the space of not controllable data is only one dimensional.*

Then, arguing by contradiction as in the proof of Proposition 15, we obtain the following observability Proposition which directly implies Theorem 3 by the HUM method (a variant with $\mathbb{R}$ linear vector space).

Proposition 19. *Let Ω be an open subet of $\mathbb{R}^2$, Γ be an open subset of $\partial\Omega$ and $0 < \widetilde{T} < T < \infty$. If the Schrödinger equation on Ω is weakly observable on $(0,\widetilde{T})\times\Gamma$ then, there exists $\mathcal{C}_1 > 0$ such that*

$$\|\phi_0\|_{H^1_0(\Omega)} \leqslant \mathcal{C}_1 \|\Im(\partial_\nu\phi)\|_{L^2((0,T)\times\Gamma)}, \quad \forall \phi_0 \in H^1_0(\Omega,\mathbb{C}) \tag{49}$$

where $\phi(t) := e^{it\Delta_D}\phi_0$.

7 Nonlinear equation on a rectangle

The goal of this section is to prove Theorem 4. Note that system (4) may be written

$$\begin{cases} i\partial_t\psi = -\Delta\psi + (-\Delta_N+1)^{-1}[\epsilon|\psi|^2]\psi + v\psi\,, & (t,x)\in(0,T)\times\Omega\,,\\ \partial_\nu\psi(t,x) = 0\,, & (t,x)\in(0,T)\times\partial\Omega\,,\\ \psi(0,x) = \psi_0(x)\,, & x\in\Omega\,,\\ (-\Delta+1)v = 0\,, & (t,x)\in(0,T)\times\Omega\,,\\ \partial_\nu v(t,x) = g(t,x)1_{\Gamma_c}(x)\,, & (t,x)\in(0,T)\times\partial\Omega\,. \end{cases} \tag{50}$$

In Section 7.1, we prove a smoothing effect. In Section 7.2, we prove the well posedness of system (50) in functional spaces appropriate for the controllability problem. In Section 7.3, we prove the C^1-regularity of the end point map. In Section 7.4, we prove the controllability of the linearized system. In Section 7.5, we prove Theorem 4 by applying the inverse mapping theorem.

7.1 Smoothing effect on the source term

We introduce the operator $\mathcal{A}$ defined by

$$D(\mathcal{A}) := H^2_N(\Omega,\mathbb{C})\,, \qquad \mathcal{A}\phi := -\Delta\phi + \frac{2\epsilon}{\pi L}(-\Delta_N+1)^{-1}[\Re(\phi)]\,.$$

On $H^2_N(\Omega,\mathbb{C})$, we use the norm

$$\|\varphi\|_{H^2_N} := \|(-\Delta+1)\varphi\|_{L^2(\Omega)} = \left(\sum_{p,n\in\mathbb{N}} \left| \left(p^2 + \left(\frac{n\pi}{L}\right)^2 + 1\right) \langle \varphi, \xi_{p,n}\rangle \right|^2 \right)^{1/2}$$

where $\langle .,.\rangle$ is the usual $L^2(\Omega)$-scalar product and

$$\xi_{p,n}(x_1,x_2) = \zeta_p(x_1)\phi_n(x_2) \tag{51}$$

$$\zeta_p(x_1) = \begin{cases} \frac{1}{\sqrt{\pi}} \text{ if } p=0\,, \\ \sqrt{\frac{2}{\pi}}\cos(px_1) \text{ if } p\in\mathbb{N}^*\,, \end{cases} \qquad \phi_n(x_2) = \begin{cases} \frac{1}{\sqrt{L}} \text{ if } n=0\,, \\ \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x_2}{L}\right) \text{ if } n\neq 0\,. \end{cases}$$

The goal of this section is the proof of the following result.

Theorem 7. *Let $T>0$, $L>0$, $\Omega=(0,\pi)\times(0,L)$ and Γ_c be an open subset of $\partial\Omega$ such that $\overline{\Gamma_c}$ does not contain any vertex of Ω. For every $\psi_0\in H^2_N(\Omega)$, $\mu_1\in L^2((0,T),H^{3/2}(\Omega))$ and $\mu_2\in L^1((0,T),H^2_N(\Omega))$ such that*

$$(-\Delta+1)\mu_1 \equiv 0\,, \quad \partial_\nu\mu_1 \in L^2((0,T)\times\partial\Omega)\,, \quad \mathit{Supp}(\partial_\nu\mu_1)\subset\Gamma_c\,, \tag{52}$$

the solution of

$$\begin{cases} \Big(i\partial_t - \mathcal{A}\Big)\psi(t,x) = \Big(\mu_1+\mu_2\Big)(t,x)\,, & (t,x)\in(0,T)\times\Omega\,, \\ \partial_\nu\psi(t,x)=0\,, & (t,x)\in(0,T)\times\partial\Omega\,, \\ \psi(0,x)=\psi_0(x)\,, & x\in\Omega\,, \end{cases}$$

satisfies $\psi\in C^0([0,T],H^2_N(\Omega))$. Moreover, there exists a constant $C>0$ (independent of ψ_0,μ_1,μ_2) such that

$$\|\psi\|_{L^0([0,T],H^2_N(\Omega))} \leqslant C\Big(\|\psi_0\|_{H^2_N(\Omega)} + \|\partial_\nu\mu_1\|_{L^2((0,T)\times\Gamma_c)} + \|\mu_2\|_{L^1((0,T),H^2_N(\Omega))}\Big)\,. \tag{53}$$

Remark 4. *By Proposition 7, $\partial_\nu\mu_1|_{\Gamma_j}$ is well defined in $L^2((0,T),H^{-\frac{3}{2}-\epsilon}(\Gamma_j))$, for $j=1,...,4$. This gives a sense to the second assumption in (52) i.e. $\partial_\nu\mu_1|_{\Gamma_j}\in L^2((0,T)\times\Gamma_j)$ for $j=1,...,4$.*

To prove Theorem 7, we need the following preliminary result.

Proposition 20. *Under the assumptions of the previous statement, the map*

$$\mathcal{H} : t \mapsto \int_0^t e^{-i\Delta_N s}\mu_1(s)ds$$

belongs to $C^0([0,T],H^2_N(\Omega))$ and

$$\|\mathcal{H}\|_{L^\infty((0,T),H^2_N(\Omega))} \leqslant C\|\partial_\nu\mu_1\|_{L^2((0,T)\times\Gamma_c)}\,, \tag{54}$$

for some constant $C=C(T)>0$ independent of μ_1.

Proof of Proposition 20: The proof is quite close to the one of Theorem 5 on the rectangle. We only perform Step 1, since the Step 2 of regularization is very similar. Actually, following similar arguments as Step 2, we can first assume $\partial_\nu \mu_{1|\partial\Omega} \in C_0^\infty((0,T)\times\Gamma_c)$ which gives by elliptic regularity $\mu_1 \in C^\infty((0,T)\times\Omega)$. The inequality (54) that we obtain for smooth functions will then allow us to get the same result under the above regularity assumptions.

So, up to now, we assume that μ_1 is regular enough.

By linearity, we may assume that $\Gamma_c \subset (a,b)\times\{0\}$ with $0<a<b<\pi$. We have

$$\begin{aligned} S = \|\mathcal{H}(t)\|^2_{H^2_N(\Omega)} &= \sum_{p,n\in\mathbb{N}} \left| \left(p^2 + \left(\tfrac{n\pi}{L}\right)^2 + 1\right) \int_0^t \langle \mu_1(s), \xi_{p,n}\rangle e^{-i\left(p^2+\left(\frac{n\pi}{L}\right)^2\right)s} ds \right|^2 \\ &= \sum_{p,n\in\mathbb{N}} \left| \int_0^t \langle \mu_1(s), (-\Delta+1)\xi_{p,n}\rangle e^{-i\left(p^2+\left(\frac{n\pi}{L}\right)^2\right)s} ds \right|^2 \end{aligned}$$

We can apply the Green formula because μ_1 and $\xi_{p,n}$ are smooth. This leads to

$$\begin{aligned} S &= \sum_{p,n=0}^{\infty} \left| \int_0^t e^{-i\left(p^2+\left(\frac{n\pi}{L}\right)^2\right)s} \int_{\Gamma_c} \partial_\nu \mu_1(s,x)\xi_{p,n}(x)d\sigma(x)ds \right|^2 \\ &\leqslant C\sum_{p=0}^{\infty}\sum_{n=0}^{\infty} \left| \int_0^{T_0} 1_{(0,t)}(s) e^{-ip^2 s} \int_0^\pi \partial_{x_2}\mu_1(s,x_1,0)\cos(px_1)dx_1 e^{-i\left(\frac{n\pi}{L}\right)^2 s} ds \right|^2 \\ &\leqslant C\sum_{p=0}^{\infty} \int_0^{T_0} \left| 1_{(0,t)}(s) e^{-ip^2 s} \int_0^\pi \partial_{x_2}\mu_1(s,x_1,0)\cos(px_1)dx_1 \right|^2 ds \end{aligned}$$

thanks to the Bessel Parseval inequality in $L^2(0,T_0)$ with $T_0 \geqslant T$ of the form $\frac{2mL^2}{\pi}$ with $m\in\mathbb{N}^*$. Thus, we have

$$\begin{aligned} S &\leqslant C\sum_{p=0}^{\infty}\int_0^t \left| \int_0^\pi \partial_{x_2}\mu_1(s,x_1,0)\cos(px_1)dx_1 \right|^2 ds \\ &\leqslant C\int_0^t \sum_{p=0}^{\infty} \left| \int_0^\pi \partial_{x_2}\mu_1(s,x_1,0)\cos(px_1)dx_1 \right|^2 ds \\ &\leqslant C\int_0^t \|\partial_{x_2}\mu_1(s,.,0)\|^2_{L^2(0,\pi)} ds \end{aligned}$$

thanks to the orthogonality of the functions $(\cos(px_1))_{p\in\mathbb{N}}$ in $L^2(0,\pi)$.

Note that μ_1 can be computed explicitly in terms of $\partial_\nu\mu_1|_{\Gamma_c}$ thanks to the rectangular form of the domain Ω. This explicit expression shows the existence of a constant $C=C(\Omega)>0$ such that

$$\|\mu_1\|^2_{L^2((0,t),H^1(\Omega))} \leqslant C\|\partial_\nu\mu_1\|_{L^2((0,t)\times\partial\Gamma_c)}\,, \quad \forall t\in(0,T)\,.$$

Therefore,

$$\|\mathcal{H}(t)\|^2_{H^2_N(\Omega)} \leqslant C(T)\|\partial_\nu\mu_1\|_{L^2((0,t)\times\partial\Gamma_c)}\,, \quad \forall t\in(0,T)\,.$$

This inequality proves that the map $\mathcal{H}$ takes values in $H^2_N(\Omega)$ on $[0,T]$ and that $\mathcal{H}:[0,T]\to H^2_N(\Omega)$ is continuous at $t=0$. The same proof shows that $\mathcal{H}$ is continuous at any $t\in(0,T]$. $\square$

Proof of Theorem 7: By the Duhamel formula, we have

$$\psi(t) = e^{i\Delta_N t}\psi_0 - i\int_0^t e^{i\Delta_N(t-s)}\Big(\widetilde{\mu}_2(s) + \mu_1(s)\Big)ds\,, \tag{55}$$

where

$$\widetilde{\mu}_2(t) := \mu_2(t) - \frac{2\epsilon}{\pi L}(-\Delta_N + 1)^{-1}(\Re\psi)\,.$$

In particular, $\widetilde{\mu}_2 \in L^1((0,T), H^2_N(\Omega))$ and

$$\|\widetilde{\mu}_2\|_{L^1((0,T),H^2_N(\Omega))} \leqslant \|\mu_2\|_{L^1((0,T),H^2_N(\Omega))} + \frac{2\epsilon T}{\pi L}\|\psi\|_{L^\infty((0,T),L^2(\Omega))}\,. \tag{56}$$

For every $\tau \in \mathbb{R}$, the operator $e^{i\Delta_N\tau}$ is an isometry of $H^2_N(\Omega)$ thus the first 2 terms in the right hand side of (55) belong to $C^0([0,T], H^2_N(\Omega))$. The third term also does, by Proposition 20, thus $\psi \in C^0([0,T], H^2_N(\Omega))$.

Step 1: Proof of (53) when T is small enough so that

$$\frac{2\epsilon T}{\pi L} \leqslant \frac{1}{2}\,. \tag{57}$$

We deduce from (55) that, for every $t \in (0,T)$

$$\|\psi(t)\|_{H^2_N(\Omega)} \leqslant \|\psi_0\|_{H^2_N(\Omega)} + \int_0^t \|\widetilde{\mu}_2(s)\|_{H^2_N(\Omega)}ds + \left\|\int_0^t e^{-i\Delta_N s}\mu_1(s)ds\right\|_{H^2_N(\Omega)}\,.$$

Using (56), Proposition 20 and (57) we obtain

$$\|\psi\|_{L^\infty((0,T),H^2_N(\Omega))} \leqslant 2\left(\|\psi_0\|_{H^2_N(\Omega)} + \|\mu_2\|_{L^1((0,T),H^2_N(\Omega))} + C\|\partial_\nu\mu_1\|_{L^2((0,T)\times\Gamma_c)}\right)\,.$$

Step 2: Proof of (53) when T is arbitrary. Let $0 = T_1 < T_2 < \ldots < T_m = T$ be a subdivision of $[0,T]$ such that $\frac{2\epsilon(T_{k+1}-T_k)}{\pi L} \leqslant \frac{1}{2}$ for $k = 1, \ldots m-1$. Applying Step 1 on each subinterval (T_k, T_{k+1}) we obtain (53). □

7.2 Wellposedness of the nonlinear Schrödinger-Poisson system

The goal of this section is the proof of the following result.

Proposition 21. *Let $T > 0$. There exists $\rho, \delta > 0$ such that, for every $\psi_0 \in H^2_N(\Omega)$, $g \in L^2((0,T)\times\partial\Omega)$ with $\|\psi_0 - \psi_{ref}(0)\|_{H^2_N} < \rho$ and $\|g\|_{L^2((0,T)\times\partial\Omega)} < \delta$, there exists a unique solution $\psi \in C^0([0,T], H^2_N(\Omega))$ of (50). Moreover $\|\psi(t)\|_{L^2(\Omega)} = \|\psi_0\|_{L^2(\Omega)}$ for every $t \in [0,T]$.*

We search ψ in the form

$$\psi(t,x,y) = \frac{e^{-\frac{i\epsilon t}{\pi L}}}{\sqrt{\pi L}}\left[1 + \zeta(t,x,y)\right]$$

where ζ is the solution of

$$\begin{cases} \Big(i\partial_t - \mathcal{A}\Big)\zeta = \Phi(\zeta) + v(1+\zeta)\,, & (t,x) \in (0,T)\times\Omega\,, \\ \partial_\nu \zeta(t,x) = 0\,, & (t,x)\in(0,T)\times\partial\Omega\,, \\ \zeta(0,x) = \zeta_0(x)\,, & x\in\Omega\,, \\ (-\Delta+1)v(t,x) = 0\,, & (t,x)\in(0,T)\times\Omega\,, \\ \partial_\nu v(t,x) = g(t,x)1_\Gamma(x)\,, & (t,x)\in(0,T)\times\partial\Omega\,, \end{cases} \tag{58}$$

and

$$\Phi(\zeta) := \frac{2\epsilon}{\pi L}(-\Delta_N+1)^{-1}\left[\Re(\zeta)\right]\zeta + \frac{\epsilon}{\pi L}(-\Delta_N+1)^{-1}\left[|\zeta|^2\right](1+\zeta)\,. \tag{59}$$

Proposition 21 is a consequence of the following result.

Proposition 22. *Let $T>0$. There exists $\rho,\delta>0$ such that, for every $\zeta_0 \in H^2_N(\Omega)$, $g\in L^2((0,T)\times\partial\Omega,\mathbb{R})$ with $\|\zeta_0\|_{H^2_N} < \rho$ and $\|g\|_{L^2((0,T)\times\partial\Omega)} < \delta$, there exists a unique weak solution $\zeta \in C^0([0,T],H^2_N(\Omega))$ of (58).*

Proof of Proposition 22: Let $T>0$.

Step 1: We prove the existence of $C_0>0$ such that, for every $\zeta\in H^2_N(\Omega)$ and $v\in H^{\frac{3}{2}}(\Omega)$ satisfying $(-\Delta+1)v=0$ in Ω, then $(-\Delta+v)(v\zeta)\in L^2(\Omega)$ and

$$\|(-\Delta+v)(v\zeta)\|_{L^2(\Omega)} \leqslant C_0\|v\|_{H^{\frac{3}{2}}(\Omega)}\|\zeta\|_{H^2_N(\Omega)}\,.$$

We have

$$(-\Delta+1)[v\zeta] = -2\nabla v\cdot\nabla\zeta - v\Delta\zeta\,.$$

- We have $\Delta\zeta\in L^2(\Omega)$ and $v\in L^\infty(\Omega)$, by the Sobolev embedding (11), thus $v\Delta\zeta\in L^2(\Omega)$.
- We have $\nabla v \in H^{1/2}(\Omega)$ and $\nabla\zeta\in H^1(\Omega)$. >From the Sobolev embedding (12), we deduce that ∇v and $\nabla\zeta$ belong to $L^4(\Omega)$. Thus $\nabla v\cdot\nabla\zeta\in L^2(\Omega)$.

Step 2: Choice of R,δ,ρ. Let $C>0$ be as in Proposition 20 and $C'>0$ be the constant of the third statement of Proposition 7. Let $C_1,C_2,C_3>0$ be such that

$$\|\Phi(\zeta)\|_{H^2_N(\Omega)} \leqslant C_1\|\zeta\|^2_{H^2_N(\Omega)}\,,\quad \forall\zeta\in H^2_N(\Omega) \text{ such that } \|\zeta\|_{H^2_N(\Omega)}\leqslant 1\,,$$

$$\|\zeta\|_{L^\infty(\overline{\Omega})} \leqslant C_2\|\zeta\|_{H^2_N(\Omega)}\,,\quad \forall\zeta\in H^2_N(\Omega)\,,$$

$$\|\Phi(\zeta)-\Phi(\widetilde{\zeta})\|_{H^2_N(\Omega)} \leqslant C_3\|\zeta-\widetilde{\zeta}\|_{H^2_N(\Omega)}\max\left\{\|\zeta\|_{H^2_N(\Omega)};\|\widetilde{\zeta}\|_{H^2_N(\Omega)}\right\}\,,$$
$$\forall\zeta\,,\widetilde{\zeta}\in H^2_N(\Omega) \text{ such that } \|\zeta\|_{H^2_N(\Omega)}\,,\|\widetilde{\zeta}\|_{H^2_N(\Omega)}\leqslant 1\,,$$

where $\Phi(\zeta)$ is defined in (59). Let

$$\begin{array}{c} R := \min\left\{1;\frac{1}{4C_1T};\frac{1}{4C_3T}\right\}\,, \qquad \rho := \frac{R}{2} \\ \delta := \min\left\{\frac{R}{4[C(1+C_2)+C_0C'\sqrt{T}]};\frac{1}{4(CC_2+C_0C'T)}\right\}\,, \end{array} \tag{60}$$

and $\zeta_0\in H^2_N(\Omega)$ be such that $\|\zeta_0\|_{H^2_N(\Omega)}<\rho$.

We consider the map

$$\left| \begin{array}{lccc} F: & \overline{B_R}[C^0([0,T],H^2_N(\Omega))] & \to & \overline{B_R}[C^0([0,T],H^2_N(\Omega))] \\ & \zeta & \mapsto & \xi \end{array} \right.$$

where $\xi := F(\zeta)$ is the solution of

$$\left\{ \begin{array}{ll} \Big(i\partial_t - \mathcal{A}\Big)\xi = \Phi(\zeta) + v(1+\zeta)\,, & (t,x,y) \in (0,T)\times\Omega\,, \\ \partial_\nu \xi(t,x,y) = 0\,, & (t,x,y) \in (0,T)\times\partial\Omega\,, \\ \xi(0,x,y) = \zeta_0(x,y)\,, & (x,y)\in\Omega\,. \end{array} \right.$$

The end of the proof consists in applying the Banach fixed point theorem to the map F.

Step 3: We prove that F takes values in $\overline{B_R}[C^0([0,T],H^2_N)]$.
Let $\zeta \in \overline{B_R}[C^0([0,T],H^2_N)]$. We introduce the solution μ_3 of

$$\left\{ \begin{array}{ll} (-\Delta+1)\mu_3(t,x_1,x_2) = (-\Delta+1)[v\zeta](t,x_1,x_2)\,, & (t,x_1,x_2)\in(0,T)\times\Omega\,, \\ \partial_\nu\mu_3(t,x_1,x_2) = 0\,, & (t,x_1,x_2)\in(0,T)\times\partial\Omega \end{array} \right.$$

and the functions

$$\mu_1 := v + v\zeta - \mu_3\,, \qquad \mu_2 := \mu_3 + \Phi(\zeta)\,.$$

By Step 1, $(-\Delta+1)(v\zeta) \in L^2((0,T)\times\Omega)$ and thus $\mu_3 \in L^2((0,T),H^2_N(\Omega))$. Therefore μ_1 and μ_2 satisfy the assumptions of Theorem 7 and $\xi := F(\zeta) \in C^0([0,T],H^2_N(\Omega))$. By Proposition 20 and (60)

$$\begin{array}{rl} \|\xi\|_{L^\infty((0,T),H^2_N)} & \leqslant \|\zeta_0\|_{H^2_N} + C\|\partial_\nu\mu_1\|_{L^2((0,T)\times\Gamma_c)} + \int_0^T \Big(\|\mu_3(s)\|_{H^2_N} + \|\Phi[\zeta(s)]\|_{H^2_N} \Big) ds \\ & \leqslant \frac{R}{2} + C\|g(1+\zeta)\|_{L^2((0,T)\times\Gamma_c)} + \sqrt{T}\|(-\Delta+1)[v\zeta]\|_{L^2((0,T)\times\Omega)} + TC_1R^2 \\ & \leqslant \frac{3R}{4} + C\delta(1+C_2) + \sqrt{T}C_0C'\delta R \\ & \leqslant R\,. \end{array}$$

Step 4: We prove that F is a contraction. Let $\zeta\,,\widetilde{\zeta} \in \overline{B_R}[C^0([0,T],H^2_N)]$. By Proposition 20 and (60),

$$\begin{array}{rl} & \|\xi - \widetilde{\xi}\|_{L^\infty((0,T),H^2_N)} \\ \leqslant & C\|\partial_\nu(\mu_1 - \widetilde{\mu}_1)\|_{L^2((0,T)\times\Gamma_c)} + \int_0^T \Big(\|(\mu_3 - \widetilde{\mu}_3)(s)\|_{H^2_N} + \|\Phi(\zeta(s)) - \Phi(\widetilde{\zeta}(s))\|_{H^2_N} \Big) ds \\ \leqslant & C\|g(\zeta - \widetilde{\zeta})\|_{L^2((0,T)\times\Gamma_c)} + \sqrt{T}\|(-\Delta+1)[v(\zeta-\widetilde{\zeta})]\|_{L^2((0,T)\times\Omega)} + TC_3R\|\zeta - \widetilde{\zeta}\|_{L^\infty((0,T),H^2_N)} \\ \leqslant & \Big(C\delta C_2 + C_0C'\delta + TC_3R \Big) \|\zeta - \widetilde{\zeta}\|_{L^\infty((0,T),H^2_N)} \\ \leqslant & \frac{1}{2}\|\zeta - \widetilde{\zeta}\|_{L^\infty((0,T),H^2_N)}\,. \end{array}$$

□

7.3 C^1-regularity of the end-point map

The end point map is defined by

$$\left| \begin{array}{llcllcl} \Theta_T: & [\mathcal{S}\cap H^2_N(\Omega,\mathbb{C})] & \times & L^2((0,T)\times\partial\Omega,\mathbb{R}) & \to & [\mathcal{S}\cap H^2_N(\Omega,\mathbb{C})]^2 \\ & (\psi_0 & , & g) & \mapsto & (\psi(0),\psi(T))\,, \end{array} \right. \qquad (61)$$

where ψ is the solution of (50). The goal of this section is to state the following result, which is a consequence of estimate (53).

Proposition 23. *Let $T > 0$ and $\rho\,, \delta > 0$ be as in Proposition 21. The end-point map Θ_T defined by (61) is C^1 on*

$$\{(\psi_0, g) \in H^2_N \times L^2((0,T)\times\partial\Omega);\, \|\psi_0 - \psi_{ref}(0)\|_{H^2_N} < \rho \text{ and } \|g\|_{L^2((0,T)\times\partial\Omega)} < \delta\}$$

and

$$d\Theta_T(\psi_{ref}(0), 0).(\Psi_0, G) = \left(\Psi_0, \frac{e^{-i\frac{\epsilon T}{\sqrt{\pi L}}}}{\sqrt{\pi L}}\Psi(T)\right)$$

where Ψ is the solution of

$$\begin{cases} \Big(i\partial_t - \mathcal{A}\Big)\Psi = V\,, & (t,x) \in (0,T)\times\Omega\,, \\ \partial_\nu \Psi(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \\ \Psi(0,x,y) = \Psi_0(x)\,, & x \in \Omega\,, \\ (-\Delta + 1)V = 0\,, & (t,x) \in (0,T)\times\Omega\,, \\ \partial_\nu V(t,x) = G(t,x)1_{\Gamma_c}(x)\,, & (t,x) \in (0,T)\times\partial\Omega\,. \end{cases} \tag{62}$$

7.4 Controllability of the linearized system

Let

$$T_{\mathcal{S}} := \left\{\varphi \in L^2(\Omega,\mathbb{C}); \Re\left(\int_\Omega \varphi(x,y)dxdy\right) = 0\right\}.$$

The goal of this section is the proof of the following result.

Proposition 24. *Let $T > 0$. There exists a continuous map*

$$\left| \begin{array}{cccc} L: & \Big(T_{\mathcal{S}} \cap H^2_N(\Omega,\mathbb{C})\Big)^2 & \to & L^2((0,T)\times\partial\Omega) \\ & (\Psi_0, \Psi_f) & \mapsto & G \end{array}\right.$$

such that, for every $(\Psi_0,\Psi_f) \in \Big(T_{\mathcal{S}} \cap H^2_N(\Omega,\mathbb{C})\Big)^2$ the solution of (62) satisfies $\Psi(T) = \Psi_f$.

7.4.1 Hilbert Uniqueness Method

The proof of Proposition 24 requires the following observability result, where $\mathcal{B}$ is defined by

$$D(\mathcal{B}) := H^2_N(\Omega,\mathbb{C})\,, \qquad \mathcal{B}\phi := -\Delta\phi + i\frac{2\epsilon}{\pi L}(-\Delta_N + 1)^{-1}[\Im(\phi)]\,.$$

Proposition 25. *Let $T > 0$. There exists $\mathcal{C} > 0$ such that, for every $\phi_T \in L^2(\Omega)$, the solution of*

$$\begin{cases} \Big(i\partial_t - \mathcal{B}\Big)\phi = 0\,, & (t,x) \in (0,T)\times\Omega\,, \\ \partial_\nu\phi(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \\ \phi(T,x) = \phi_T(x)\,, & x \in \Omega\,, \end{cases} \tag{63}$$

satisfies

$$\|\phi_T\|_{L^2(\Omega)} \leqslant \mathcal{C}\|\Im(\phi)\|_{L^2((0,T)\times\Gamma_c)}\,. \tag{64}$$

Note that the solution of (63) can be computed explicitly (as in the case $\epsilon = 0$) because $\mathcal{B}$ preserves $\mathbb{C}\xi_{p,n}$. The boundary condition in (63) has to be understood in the sense of the semi-group. The proof of Proposition 25 relies on 2 intermediate results:

- the weak observability of (63), proved in Section 7.4.2 below,
- a unique continuation property, proved in Section 7.4.3.

Note that the trace $\Im(\phi)|_{\partial\Omega}$ is well defined in $L^2((0,T)\times\partial\Omega)$ (see Lemma 1 below).

Proof of Proposition 24 thanks to Proposition 25: We want to prove the existence of a continuous right inverse for the continuous operator

$$\left| \begin{array}{rcl} F: \quad L^2((0,T)\times\Gamma_c,\mathbb{R}) & \to & T_{\mathcal{S}} \\ G & \mapsto & (-\Delta+1)\Psi(T) \end{array} \right.$$

where Ψ is the solution of (62) with $\Psi_0 = 0$. By [12, Theorems II.10 and II.19], it is equivalent to prove the existence of a constant $\mathcal{C} > 0$ such that

$$\|\phi_T\|_{L^2(\Omega)} \leqslant \mathcal{C}\|F^*(\phi_T)\|_{L^2((0,T)\times\Gamma_c)}\,, \quad \forall \phi_T \in T_{\mathcal{S}}\,.$$

Thus, to deduce Proposition 24 from Proposition 25, it is sufficient to prove that

$$F^*(\phi_T) = -\Im(\phi)|_{\Gamma_c}\,, \quad \forall \phi_T \in T_{\mathcal{S}}\,,$$

where ϕ is associated to ϕ_T by (63). The trace $\Im(\phi)|_{\Gamma_c}$ is defined as an extension, thus, it is sufficient to prove that

$$F^*(\phi_T) = -\Im(\phi)|_{\Gamma_c}\,, \quad \forall \phi_T \in T_{\mathcal{S}} \cap C_c^\infty(\Omega)\,. \tag{65}$$

Let $\phi_T \in T_{\mathcal{S}} \cap C_c^\infty(\Omega)$. Then $\phi \in C^\infty([0,T], H^s_N(\Omega))$ for every $s > 0$, thus $\widetilde{\phi} := (-\Delta+1)\phi \in C^\infty([0,T], H^s_N(\Omega))$ for every $s > 0$ and

$$\left\{ \begin{array}{ll} \Big(i\partial_t + \mathcal{B}\Big)\widetilde{\phi} = 0\,, & (t,x) \in (0,T)\times\Omega\,, \\ \partial_\nu \widetilde{\phi}(t,x) = 0\,, & (t,x) \in (0,T)\times\partial\Omega\,, \\ \widetilde{\phi}(T,x) = (-\Delta+1)\phi_T(x)\,, & x \in \Omega\,, \end{array} \right.$$

because $\mathcal{B}$ and $(-\Delta+1)$ commute. The functions Ψ and $\widetilde{\phi}$ are solutions in the sense of the semi-group, thus

$$\begin{array}{rl} \Re\langle \Psi(T), \widetilde{\phi}(T)\rangle & = \Im\left(\int_0^T \int_\Omega V(t,x)\overline{\widetilde{\phi}(t,x)}dxdt\right) \\ & = \Im\left(\int_0^T \int_\Omega V(t,x)\overline{(-\Delta+1)\phi(t,x)}dxdt\right) \end{array}$$

Therefore,

$$\begin{array}{rl} \Re\langle F(G), \phi_T\rangle & = \Re\langle(-\Delta+1)\Psi(T), \phi_T\rangle \\ & = \Re\langle \Psi(T), (-\Delta+1)\phi_T\rangle \text{ because } \Psi(T)\,, \phi_T \in H^2_N(\Omega) \\ & = \Im\left(\int_0^T \int_\Omega V(t,x)\overline{(-\Delta+1)\phi(t,x)}dxdt\right) \\ & = -\int_0^T \int_{\Gamma_c} G(t,x)\Im[\phi(t,x)]d\sigma(x)dt\,. \end{array}$$

This proves (65). □

Lemma 1. *Let $T > 0$. There exists $C = C(T) > 0$ such that, for every $\phi_0 \in L^2(\Omega)$, the solution $\phi \in C^0([0,T], L^2(\Omega))$ of (63) satisfies*

$$\int_0^T \int_{\partial\Omega} |\phi(t,x)|^2 d\sigma(x) dt \leqslant C \|\phi_0\|^2_{L^2(\Omega)} .$$

Proof of Lemma 1:

Step 1: Proof when $\epsilon = 0$. The following equality holds in $C^0([0,T], L^2(\Omega))$

$$\phi(t, x_1, x_2) = \sum_{p,n=0}^{\infty} \langle \phi_0, \xi_{p,n} \rangle e^{-i\left[p^2 + \left(\frac{n\pi}{L}\right)^2\right]t} \xi_{p,n}(x_1, x_2) .$$

Using (51) and the orthogonality of $(\zeta_p)_{p\in\mathbb{N}}$ in $L^2(0,\pi)$, we get

$$\begin{aligned}
\int_0^T \int_0^\pi |\phi(t,x_1,0)|^2 dx_1 dt &= \int_0^T \int_0^\pi \left| \sum_{p=0}^{\infty} \left(\sum_{n=0}^{\infty} \langle \phi_0, \xi_{p,n} \rangle e^{-i\left[p^2 + \left(\frac{n\pi}{L}\right)^2\right]t} \right) \zeta_p(x_1) \right|^2 dx_1 dt \\
&= \int_0^T \sum_{p=0}^{\infty} \left| \sum_{n=0}^{\infty} \langle \phi_0, \xi_{p,n} \rangle e^{-i\left[p^2 + \left(\frac{n\pi}{L}\right)^2\right]t} \right|^2 dt \\
&= \sum_{p=0}^{\infty} \int_0^T \left| \sum_{n=0}^{\infty} \langle \phi_0, \xi_{p,n} \rangle e^{-i\left(\frac{n\pi}{L}\right)^2 t} \right|^2 dt \\
&\leqslant \sum_{p=0}^{\infty} C_{\text{Ing}} \sum_{n=0}^{\infty} |\langle \phi_0, \xi_{p,n} \rangle|^2 = C_{\text{Ing}} \|\phi_0\|^2_{L^2(\Omega)} ,
\end{aligned}$$

where $C_{\text{Ing}}(T) > 0$ is the Ingham constant associated to the familly $\left(e^{-i\left(\frac{n\pi}{L}\right)^2 t}\right)_{n\in\mathbb{N}}$ in $L^2(0,T)$ (see [19]).

Step 2: Proof when $\epsilon \neq 0$. The following equality holds in $L^2(\Omega)$.

$$\phi(t) = e^{-i\Delta_N t}\phi_0 + \frac{2\epsilon}{\pi L} \int_0^t e^{-i\Delta_N (t-s)} (-\Delta_N + 1)^{-1} [\Im(\phi(s))] ds , \quad \forall t \in [0,T] .$$

By step 1, the first term of the right hand side has a trace on $\partial\Omega$ with an $L^2((0,T)\times\partial\Omega)$-norm bounded by $\sqrt{C_{Ing}}\|\phi_0\|_{L^2(\Omega)}$. The second term of the right hand side belongs to $C^0([0,T], H^2_N(\Omega))$ and its $L^\infty((0,T), H^2_N(\Omega))$-norm is bounded by

$$\frac{2\epsilon}{\pi L} \int_0^T \|\phi(s)\|_{L^2(\Omega)} = \frac{2\epsilon T}{\pi L} \|\phi_0\|_{L^2(\Omega)} .$$

Thus, this term as a trace on $\partial\Omega$ with an $L^2((0,T)\times\partial\Omega)$-norm that satisfies a similar estimate. □

7.4.2 Weak observability

The goal of this section is the proof of the following result.

Proposition 26. *Let $T > 0$. There exists $\mathcal{C}' = \mathcal{C}'(T) > 0$ such that, for every $\phi_T \in L^2(\Omega)$, the solution of (63) satisfies*

$$\|\phi_T\|_{L^2(\Omega)} \leqslant \mathcal{C}' \Big(\|\Im(\phi)\|_{L^2((0,T)\times\Gamma_c)} + \|\phi_T\|_{H^{-2}(\Omega)} \Big) . \tag{66}$$

Proof of Proposition 26: First, we recall Tenenbaum and Tucsnak's result: for every $T>0$ and every nonempty open subset Γ of $\partial\Omega$, there exists $\mathcal{C}_0 = \mathcal{C}_0(T,\Gamma)>0$ such that, for every $\phi_T \in L^2(\Omega)$, the solution $\widetilde{\phi}$ of

$$\left\{\begin{array}{ll} \Big(i\partial_t + \Delta\Big)\widetilde{\phi}(t,x) = 0\,, & (t,x)\in(0,T)\times\Omega\,,\\ \partial_\nu \widetilde{\phi}(t,x) = 0\,, & (t,x)\in(0,T)\times\partial\Omega\,,\\ \widetilde{\phi}(T,x) = \phi_T(x)\,, & x\in\Omega\,, \end{array}\right. \tag{67}$$

satisfies

$$\|\phi_T\|_{L^2(\Omega)} \leqslant \mathcal{C}_0 \|\widetilde{\phi}\|_{L^2((0,T)\times\Gamma)}\,. \tag{68}$$

Step 1: We prove that, for every $T>0$, there exists $\mathcal{C}_1=\mathcal{C}_1(T)>0$ such that for every $\phi_T\in L^2(\Omega)$, the solution $\widetilde{\phi}$ of (67) satisfies

$$\|\phi_T\|_{L^2(\Omega)} \leqslant \mathcal{C}_1\Big(\|\Im(\widetilde{\phi})\|_{L^2((0,T)\times\Gamma_c)} + \|\phi_T\|_{H^{-1}(\Omega)}\Big)\,. \tag{69}$$

Let $T>0$ and $\rho\in C^\infty_c(\mathbb{R},\mathbb{R}^+)$ be such that $\rho\equiv 1$ on $(T/3,2T/3)$ and $\mathrm{Supp}(\rho)\subset(0,T)$. Let $\phi_T\in L^2(\Omega)$. We have

$$\begin{array}{rl} \|\phi_T\|^2_{L^2(\Omega)} & \leqslant \mathcal{C}_0^2 \int_{T/3}^{2T/3}\int_{\Gamma_c} |\widetilde{\phi}(t,x)|^2 d\sigma(x)dt \quad \text{with } \mathcal{C}_0=\mathcal{C}_0(T/3) \text{ by (68)}\\ & \leqslant \mathcal{C}_0^2 \int_{\mathbb{R}}\int_{\Gamma_c} |\rho(t)\widetilde{\phi}(t,x)|^2 d\sigma(x)dt\\ & \leqslant \frac{\mathcal{C}_0^2}{2}\int_{\mathbb{R}}\int_{\Gamma_c}\Big(|\rho(t)\widetilde{\phi}(t,x)|^2 + |\rho(t)\overline{\widetilde{\phi}(t,x)}|^2\Big)d\sigma(x)dt\\ & \leqslant \frac{\mathcal{C}_0^2}{2}\int_{\mathbb{R}}\int_{\Gamma_c}\Big(|\rho(t)\Im(\widetilde{\phi}(t,x))|^2 + 2\Re[\rho(t)^2\widetilde{\phi}(t,x)^2]\Big)d\sigma(x)dt\,. \end{array}$$

In order to get (69), it suffices to prove the existence of $C>0$ such that

$$\int_{\mathbb{R}}\int_{\Gamma_c} \Re[\rho(t)^2\widetilde{\phi}(t,x)^2]d\sigma(x)dt \leqslant C\|\phi_T\|^2_{H^{-1}(\Omega)}\,.$$

The function ρ^2 belongs to $C^\infty_c(\mathbb{R},\mathbb{R})$, thus, for every $N\in\mathbb{N}^*$, there exists $C_N>0$ such that

$$\left|\int_{\mathbb{R}}\rho(t)^2 e^{-i\omega t}dt\right| \leqslant \frac{C_N}{(1+\omega)^N}\,, \quad \forall \omega>0\,.$$

We have

$$\begin{array}{rl} & \left|\int_{\mathbb{R}}\int_{\Gamma_c}\rho(t)^2\widetilde{\phi}(t,x)^2 d\sigma(x)dt\right|\\ = & \left|\sum\limits_{j,J\in\mathbb{N}^*}\phi_{T,j}\phi_{T,J}\left(\int_{\mathbb{R}}\rho(t)^2 e^{-(\lambda_j+\lambda_J)t}dt\right)\Big(\int_{\Gamma_c}\varphi_j(x)\varphi_J(x)d\sigma(x)\Big)\right|\\ \leqslant & \sum\limits_{j,J\in\mathbb{N}^*}|\phi_{T,j}\phi_{T,J}|\frac{C_N}{(1+\lambda_j+\lambda_J)^N}C\lambda_j^\alpha\lambda_J^\alpha \quad \text{for some } C,\alpha>0\\ \leqslant & CC_N\left(\sum\limits_{j\in\mathbb{N}^*}|\phi_{T,j}|\frac{\lambda_j^\alpha}{(1+\lambda_j)^N}\right)^2\\ \leqslant & CC_N\left(\sum\limits_{j\in\mathbb{N}^*}\frac{|\phi_{T,j}|^2}{1+\lambda_j}\right)^2\left(\sum\limits_{j\in\mathbb{N}^*}\frac{\lambda_j^{2\alpha}(1+\lambda_j)}{(1+\lambda_j)^{2N}}\right)^2\\ \leqslant & C\|\phi_T\|^2_{H^{-1}(\Omega)} \quad \text{for } N \text{ large enough.} \end{array}$$

Step 2: Conclusion. Working by contradiction, we assume the existence of a sequence $(\phi_T^n)_{n\in\mathbb{N}^*}$ such that the associated solutions ϕ^n of (63) satisfy

$$1 = \|\phi_T^n\|_{L^2(\Omega)} > n\Big(\|\Im(\phi^n)\|_{L^2((0,T)\times\Gamma_c)} + \|\phi_T^n\|_{H^{-2}(\Omega)}\Big)\,. \qquad (70)$$

We introduce the solutions $\widetilde{\phi}^n$ of (67) associated to final condition $\widetilde{\phi}^n(T) = \phi_T^n$. Then $\xi^n := \phi^n - \widetilde{\phi}^n$ solves

$$\left\{\begin{array}{ll} \Big(i\partial_t + \Delta\Big)\xi^n = i\frac{2\epsilon}{\pi L}(-\Delta_N + 1)^{-1}\Im(\phi^n)\,, & (t,x)\in(0,T)\times\Omega\,,\\ \partial_\nu \xi^n(t,x) = 0\,, & (t,x)\in(0,T)\times\partial\Omega\,,\\ \xi^n(T,x) = 0\,. & x\in\Omega\,, \end{array}\right.$$

>From (70), we deduce that $\phi_T^n \underset{n\to\infty}{\longrightarrow} 0$ in $H^{-2}(\Omega)$. Moreover this sequence is bounded in $L^2(\Omega)$, thus, by interpolation $\phi_T^n \underset{n\to\infty}{\longrightarrow} 0$ in $H^{-1}(\Omega)$. Therefore, $(-\Delta_N+1)^{-1}\Im(\phi^n) \underset{n\to\infty}{\longrightarrow} 0$ in $L^\infty((0,T),H^1(\Omega))$ and $\xi^n \underset{n\to\infty}{\longrightarrow} 0$ in $L^\infty((0,T),H^1(\Omega))$. As a consequence $\Im(\xi^n)\Big|_{\Gamma_c} \underset{n\to\infty}{\longrightarrow} 0$ in $L^\infty((0,T)\times\Gamma_c)$. Moreover, $\Im(\phi^n)\Big|_{\Gamma_c} \underset{n\to\infty}{\longrightarrow} 0$ in $L^2((0,T)\times\Gamma_c)$ by (70), thus $\Im(\widetilde{\phi^n})\Big|_{\Gamma_c} \underset{n\to\infty}{\longrightarrow} 0$ in $L^2((0,T)\times\Gamma_c)$. >From Step 1, we get

$$1 = \|\phi_T^n\|_{L^2(\Omega)} \leqslant \mathcal{C}_1\Big(\|\Im(\widetilde{\phi}^n)\|_{L^2((0,T)\times\Gamma_c)} + \|\phi_T^n\|_{H^{-1}(\Omega)}\Big) \underset{n\to\infty}{\longrightarrow} 0\,,$$

which is a contradiction. □

7.4.3 Unique continuation

The goal of this section is to prove the following result.

Proposition 27. *We assume that ϵ satisfies (6). Let $T>0$, Γ_c be an open subset of $\partial\Omega$ and $\phi_T \in T_{\mathcal{S}}\setminus\{0\}$ and ϕ be the solution of (63). Then $\Im(\phi)|_{\Gamma_c}$ does not identically vanish on $(0,T)\times\Gamma_c$.*

Proof of Proposition 27: To simplify, we assume that $\Gamma_c = (a,b)\times\{0\}$ where $0<a<b<1$. We introduce

$$N_T := \{\phi_0 \in T_{\mathcal{S}}; \Im(\phi) = 0 \text{ on } (0,T)\times\Gamma_c\}$$

where $\phi(t) := S(t)\phi_0$ solves (63) with initial condition at $t=0$: $\phi(0)=\phi_0$. N_T is a $\mathbb{R}$-vector subspace of $L^2(\Omega)$. Working by contradiction, we assume that $N_T \neq \{0\}$.

Step 1: We prove that N_T is stable by $(-i\mathcal{B})$ and has finite dimension. Let $\phi_0\in N_T$. Since $\phi_0 \in L^2(\Omega)$, then $\phi_\epsilon := \frac{S(\epsilon)\phi_0 - \phi_0}{\epsilon}$ is bounded in $H_N^{-2}(\Omega)$ when $[\epsilon\to 0]$. Moreover, for $\epsilon < T/2$, $\phi_\epsilon \in N_{T/2}$. Applying (66) to ϕ_ϵ, we get

$$\|\phi_\epsilon\|_{L^2(\Omega)} \leqslant \mathcal{C}\,(T/2)\,\|\phi_\epsilon\|_{H_N^{-2}(\Omega)}\,, \quad \forall\epsilon\in(0,T/2)\,.$$

Thus ϕ_ϵ is bounded in $L^2(\Omega)$ when $[\epsilon \to 0]$. Therefore $\phi_0 \in H^2_N(\Omega)$. Indeed, by Fatou lemma,

$$\begin{aligned}
\|\phi_0\|^2_{H^2_N(\Omega)} &= \sum_{k=1}^{\infty} |\lambda_k \langle \phi_0, \varphi_k\rangle|^2 \\
&= \sum_{k=1}^{\infty} \liminf_{\epsilon\to 0} \left| \frac{e^{-i\lambda_k \epsilon}-1}{\epsilon} \langle \phi_0, \varphi_k\rangle \right|^2 \\
&\leqslant \liminf_{\epsilon\to 0} \sum_{k=1}^{\infty} \left| \frac{e^{-i\lambda_k \epsilon}-1}{\epsilon} \langle \phi_0, \varphi_k\rangle \right|^2 \\
&\leqslant \liminf_{\epsilon\to 0} \|\phi_\epsilon\|^2_{L^2(\Omega)} < \infty\,.
\end{aligned}$$

Let $T' \in (0,T)$. The map

$$\left| \begin{array}{ccc} \mathcal{J}_{T'}: & L^2(\Omega) & \to & L^2((0,T')\times\Gamma_c) \\ & \phi_0 & \mapsto & \Im(\phi)\Big|_{\Gamma_c} \end{array} \right.$$

is continuous (see Lemma 1). Moreover, $\mathcal{J}_{T'}(\phi_\epsilon) = 0$ for every $\epsilon \in (0, T-T')$ and $\phi_\epsilon \longrightarrow -i\mathcal{B}\phi_0$ in $L^2(\Omega)$ when $[\epsilon \to 0]$ thus $\mathcal{J}_{T'}(-i\mathcal{B}\phi_0) = 0$. This holds for every $T' \in (0,T)$, therefore $-i\mathcal{B}\phi_0 \in N_T$. We have proved that N_T is stable by $-i\mathcal{B}$ and only contains smooth functions in $D[\mathcal{B}^s]$, $\forall s \in \mathbb{N}^*$. Applying estimate (66) to $(-i\mathcal{B}\phi_0)$, we get

$$\|\mathcal{B}\phi_0\|_{L^2(\Omega)} \leqslant \mathcal{C}'\|\mathcal{B}\phi_0\|_{H^{-2}(\Omega)}\,.$$

This shows that the unit ball of N_T is compact (for the $L^2(\Omega)$-topology), thus N_T has finite dimension, by Riesz theorem.

Step 2: We prove that N_T contains a finite sum of functions $\xi_{p,n}$. The finite dimensional $\mathbb{R}$-vector space N_T is stable by the operator $-i\mathcal{B}$, thus there exists $\alpha, \beta \in \mathbb{R}$ such that

$$\mathrm{Ker}[P(-i\mathcal{B})] \cap N_T \neq \{0\} \qquad \text{where } P(X) = X^2 + \alpha X + \beta\,.$$

Let $\phi_0 \in \mathrm{Ker}[P(-i\mathcal{B})] \cap N_T$. Let $a_{p,n}, b_{p,n} \in \mathbb{R}$ be such that

$$\langle \phi_0, \xi_{p,n}\rangle = a_{p,n} + i b_{p,n}\,, \quad \forall p,n \in \mathbb{N}\,.$$

Note that $a_{0,0} = 0$ because $\phi_0 \in T_{\mathcal{S}}$. Explicit computations show that the relation $P(-i\mathcal{B})\phi_0 = 0$ writes

$$\sum_{p,n=0}^{\infty} \left(P[-i\lambda_{p,n}] a_{p,n} + iP\left[-i\left(\lambda_{p,n} + \frac{2\epsilon}{\pi L(\lambda_{p,n}+1)} \right) \right] b_{p,n} \right) \xi_{p,n} = 0$$

where $\lambda_{p,n} := p^2 + \left(\frac{n\pi}{L}\right)^2$. Thus

$$P[-i\lambda_{p,n}] a_{p,n} = P\left[-i\left(\lambda_{p,n} + \frac{2\epsilon}{\pi L(\lambda_{p,n}+1)} \right) \right] b_{p,n} = 0\,, \quad \forall p,n \in \mathbb{N}\,.$$

The polynomial P has at most 2 roots and the 2 quantities $\lambda_{p,n}$ and $\lambda_{p,n} + \frac{2\epsilon}{\pi L(\lambda_{p,n}+1)}$ diverge when $\|(n,p)\| \to \infty$, thus only a finite number of coefficients $\langle \phi_0, \xi_{p,n}\rangle$ can be different from zero.

Step 3: Conclusion. Let $N \in \mathbb{N}$ be such that $\phi_0 = \sum_{n,p=0}^{N} \langle \phi_0, \xi_{p,n} \rangle \xi_{n,p}$.

Step 3.1:
Assumption (6) implies that $\lambda_{p,n} + \frac{2\epsilon}{\pi L(\lambda_{p,n}+1)} > 0$ for every $n, p \in \mathbb{N}$. Explicit computations show that

$$\Im[\phi(t, x_1, 0)] = \sum_{p,n=0}^{N} \left(b_{p,n} \cos(\omega_{p,n} t) - \gamma_{p,n} a_{p,n} \sin(\omega_{p,n} t) \rangle \right) \right) \zeta_p(x_1)$$

where

$$\omega_{p,n} := \sqrt{\lambda_{p,n} \left(\lambda_{p,n} + \frac{2\epsilon}{\pi L(\lambda_{p,n} + 1)} \right)}, \quad \forall p, n \in \mathbb{N},$$

$$\gamma_{p,n} := \sqrt{\frac{\lambda_{n,p}}{\lambda_{p,n} + \frac{2\epsilon}{\pi L(\lambda_{p,n}+1)}}}.$$

The continuous function $(t, x) \mapsto \Im[\phi(t, x_1, 0)]$ vanishes on $(0, T) \times (a, b)$ because $\phi_0 \in N_T$. In particular, for every $t \in [0, T]$, the function $x_1 \mapsto \Im[\phi(t, x_1, 0)]$ vanishes on (a, b). But the functions $(x_1 \mapsto \cos(px_1))_{0 \leqslant p \leqslant N}$ are linearly independent on (a, b), thus

$$\sum_{n=0}^{N} \left(b_{p,n} \cos(\omega_{p,n} t) - \gamma_{p,n} a_{p,n} \sin(\omega_{p,n} t) \right) = 0 \, , \forall t \in (0, T) \, , 1 \leqslant p \leqslant N \, .$$

We notice that assumption (6) on ϵ imply

$$\omega_{p,n}^2 \neq \omega_{p,m}^2 \, , \quad \forall p, n, m \in \mathbb{N} \text{ such that } n \neq m \text{ and } (p, n) \, , (p, m) \neq (0, 0)$$

where

$$\omega_{p,n}^2 := \left(p^2 + \left(\frac{n\pi}{L} \right)^2 \right) \left(p^2 + \left(\frac{n\pi}{L} \right)^2 + \frac{2\epsilon}{\pi L \left(p^2 + \frac{n\pi}{L} + 1 \right)} \right), \quad \forall p, n, m \in \mathbb{N} \, .$$

Under assumption (6), the map $f(s) := s \left(s + \frac{2\epsilon}{\pi L(s+1)} \right)$ is strictly increasing on $[m, \infty)$ and thus the above property holds.

So, the frequencies $\{\omega_{p,n}; 0 \leqslant n \leqslant N\}$ are all different for every $p \in \mathbb{N}$. Thus, the previous relations imply that $b_{p,n} = a_{p,n} = 0$ for every $0 \leqslant p, n \leqslant N$, i.e. $\phi_0 = 0$, which is a contradiction. □

7.4.4 Observability

Proposition 25 results from Propositions 26 and 27, by working as in the proof of Proposition 15.

7.5 Controllability of the nonlinear PDE

The proof of Theoorem 4 consists in applying the inverse mapping theorem to the end-point map Θ defined in (61), as in Section 4.2.

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